The Hilbert transform

Author: Aleksis Koski
Supervisors: Prof. Eero Saksman
Prof. Tadeusz Iwaniec

February 19, 2011
Contents

1 Definitions 2

2 Introduction 2

3 The Poisson kernel and harmonic conjugates 5
   3.1 The Poisson kernel . . . . . . . . . . . . . . . . . . . . . . . . 5
   3.2 Conjugate functions . . . . . . . . . . . . . . . . . . . . . . . 8

4 Continuous linear extensions 11

5 $L^1$ theory of the Hilbert transform 14

6 $L^2$ theory of the Hilbert transform 24
   6.1 The Fourier transform . . . . . . . . . . . . . . . . . . . . . . . 26

7 The Marcinkiewicz interpolation theorem 29
   7.1 Boundedness of the Hilbert transform using duality . . . . . . . 32
1 Definitions

This is a list of definitions and notation used in the following sections.

The Lebesgue measure of a set $A \subset \mathbb{R}$ will be denoted by $|A|$.

The $L^p$ spaces for $1 \leq p < \infty$ are defined to be the spaces of equivalence classes of measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ such that $|f|^p$ is integrable over $\mathbb{R}$. Here the equivalence relation is defined by $f \sim g \Leftrightarrow f(x) = g(x)$ for almost every $x$. In addition, the $L^\infty$ space is defined to be the space of equivalence classes of all essentially bounded measurable functions from $\mathbb{R}$ to $\mathbb{C}$. The $L^p$ spaces are known to be Banach spaces with respect to the norms

$$||f||_p = \left( \int_\mathbb{R} |f(x)|^p dx \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$||f||_\infty = \inf\{ C > 0 : |\{ x \in \mathbb{R} : |f(x)| > C \}| = 0 \}.$$

The characteristic function of a set $A \subset \mathbb{R}$ is defined to take the value 1 in $A$ and 0 in $\mathbb{R} \setminus A$. We will denote this function by $\chi_A$.

The support of a continuous function $f$ is defined by

$$\text{supp}(f) = \{ x : f(x) \neq 0 \}.$$

The set $C^\infty_0$ will denote the set of functions from $\mathbb{R}$ to $\mathbb{C}$ that are smooth and have compact support. The space $C^\infty_0$ is known to be dense in $L^p$ for $1 \leq p < \infty$.

2 Introduction

Assume that we are given a function $f : \mathbb{R} \mapsto \mathbb{C}$. We would like to be able to define another function, called the Hilbert transform of $f$ and denoted by $\mathcal{H}f$, by the equation

$$\mathcal{H}f(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t| > \epsilon} \frac{f(x-t)}{t} dt = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|x-t| > \epsilon} \frac{f(t)}{x-t} dt.$$

It is not clear however that such a limit exists even if we assume that $f$ is sufficiently well-behaved. The problem is that the factor $1/t$ in the integral
decays slowly for large \( t \) and has a problematic singularity at the origin. The slow decay can be averted by assuming for example that \( f \) is integrable over \( \mathbb{R} \). If \( f \) decays fast enough, then for each \( \epsilon > 0 \) we can also define the truncated Hilbert transform of \( f \), denoted by \( \mathcal{H}_\epsilon f \), with
\[
\mathcal{H}_\epsilon f(x) = \frac{1}{\pi} \int_{|t|>\epsilon} \frac{f(x-t)}{t} dt.
\]
This can also be regarded as a convolution,
\[
\mathcal{H}_\epsilon f(x) = \left( \frac{\chi_\epsilon(x)}{\pi x} * f \right)(x),
\]
where we have used the notation \( \chi_\epsilon(x) = \chi_{\{|x|>\epsilon\}}(x) \). It should be noted that this notation is used in future sections as well. The fast enough decay of \( f \) can be stated more precisely by assuming that \( f \in L^p \) for some \( p \) with \( 1 \leq p < \infty \). Hölder’s inequality tells us that the above convolution is well-defined since \( \chi_\epsilon(x)/x \in L^p \) for \( 1 < p \leq \infty \). However \( \chi_\epsilon(x)/x \notin L^1 \), and due to complications arising from this in trying to define the Hilbert transform on \( L^\infty \), we will only focus on the remaining \( L^p \)-spaces with \( 1 \leq p < \infty \) in the future.

Let us now look for cases where the Hilbert transform is clearly well-defined. Observe that the function \( 1/x \) is odd, which means that if \( f(x) \) is sufficiently regular we could hope that the positive and negative singular parts of \( 1/x \) cancel each other out. If for example \( f(x) \) is constant in a small interval around some \( x_0 \), say \([x_0-\delta,x_0+\delta]\) with \( \delta > 0 \), then the value of \( \mathcal{H} f \) at \( x_0 \) is well-defined as we may calculate
\[
\mathcal{H} f(x_0) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|>\epsilon} \frac{f(x_0-t)}{t} dt
= \frac{1}{\pi} \int_{|t|>\delta} \frac{f(x_0-t)}{t} dt
+ \lim_{\epsilon \to 0} \left( \frac{1}{\pi} \int_{\delta t>\epsilon} \frac{f(x_0-t)}{t} dt + \frac{1}{\pi} \int_{-\delta t<-\epsilon} \frac{f(x_0-t)}{t} dt \right)
= \frac{1}{\pi} \int_{|t|>\delta} \frac{f(x_0-t)}{t} dt,
\]
where the two terms inside the limit cancel due to \( f \) being constant around \( x_0 \). We now use this to prove that if \( f \) is continuously differentiable and integrable over \( \mathbb{R} \), then \( \mathcal{H} f \) is well-defined for each \( x \in \mathbb{R} \). This is because
we may write

\[ \mathcal{H} f(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t| > \epsilon} \frac{f(x - t)}{t} dt \]

\[ = \frac{1}{\pi} \int_{|t| \geq 1} \frac{f(x - t)}{t} dt + \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t| > \epsilon} \frac{f(x - t) - f(x)}{t} dt \]

\[ + \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{1 > |t| > \epsilon} \frac{f(x)}{t} dt. \]

Now each of the terms on the utmost right hand side can be seen to be well-defined. The first one because \( f \) is integrable and \( 1/t \) bounded. The last limit exists since we have a constant numerator inside and thus the singular parts cancel as noted before. And lastly for the second term we may apply the mean value theorem on the real and imaginary parts of \( f \) to show that the quantity

\[ \frac{f(x - t) - f(x)}{t} \]

is bounded on \([-1, 1]\), implying that the second term is finite.

Having shown that the Hilbert transform is defined for such a large class of functions, it would be a natural question to ask how wildly it behaves with respect to some of the common function spaces, such as \( L^p \) for \( 1 \leq p < \infty \). Note that as long as we define the Hilbert transform using formula (2.1), it defines a linear transformation since

\[ \mathcal{H} (\alpha f + \beta g) = \lim_{\epsilon \to 0} \mathcal{H}_\epsilon (\alpha f + \beta g) = \alpha \lim_{\epsilon \to 0} \mathcal{H}_\epsilon f + \beta \lim_{\epsilon \to 0} \mathcal{H}_\epsilon g = \alpha \mathcal{H} f + \beta \mathcal{H} g \]

for functions \( f, g \) and complex numbers \( \alpha, \beta \). Thus if we would like to prove the continuity of \( \mathcal{H} \) as an operator, it is enough to show that it defines a bounded linear transformation. In the case of \( L^p \) spaces, for each \( p \) we would like to find a constant \( C_p \) such that

\[ \|\mathcal{H} f\|_p \leq C_p \|f\|_p. \]

This is called the strong \((p, p)\) inequality. We haven’t defined the Hilbert transform on the whole \( L^p \) yet, but it turns out that there exists a natural definition for it such that the strong \((p, p)\) inequality is true for \( 1 < p < \infty \). For \( p = 1 \) the strong inequality fails to hold, but we will instead recover a weaker bound. The proofs of these facts are quite nontrivial and they will be explored in later sections. Instead we will first focus on showing some of the motivation behind the Hilbert transform.
3 The Poisson kernel and harmonic conjugates

In this part we will show how the definition of the Hilbert transform arises from elementary complex analysis in a very natural way. Let us first revisit some definitions:

**Definition 3.1.** Let \( \Omega \subset \mathbb{C} \) be a domain. We say that a function \( u : \Omega \to \mathbb{R} \) is harmonic if \( u \) is twice continuously differentiable and

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

There is a simple relation between harmonic and analytic functions. If an analytic function \( f \) defined in some domain \( \Omega \) is written as \( f(z) = u(z) + iv(z) \), where \( u \) and \( v \) are the real and imaginary parts of \( f \) respectively, then the Cauchy-Riemann equations imply that \( u \) and \( v \) are both harmonic in \( \Omega \). On the other hand, it is also known that if the domain \( \Omega \) is assumed to be convex, then for every harmonic function \( u \) in \( \Omega \) there exists another harmonic function \( v \) such that \( u + iv \) is analytic in \( \Omega \). Thus each harmonic function is at least locally the real part of an analytic function.

One of the main questions in the theory of harmonic functions is the so-called Dirichlet problem. If we are given a domain \( \Omega \) and a continuous function \( f : \partial \Omega \to \mathbb{R} \), the Dirichlet problem asks whether there exists a harmonic function \( u : \Omega \to \mathbb{R} \) such that \( \lim_{z \to \xi} u(z) = f(\xi) \) for all \( \xi \in \partial \Omega \). In this section we will only consider the special case where \( \Omega \) is the upper-half complex plane \( \mathbb{H} \) defined by

\[
\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}.
\]

Note that the boundary of \( \mathbb{H} \) is the real line, which means that \( f \) in this case is just a real function on \( \mathbb{R} \). It turns out that this special case of the Dirichlet problem has a simple solution for sufficiently well-behaved functions \( f \), but we will need to define a useful tool first.

### 3.1 The Poisson kernel

**Definition 3.2.** We define the **Poisson kernel** \( P(x, y) \) by

\[
P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}
\]

for all \( (x, y) \in \mathbb{H} \).
Theorem 3.1. The Poisson kernel has the following properties:

\[ P(x, y) \geq 0 \text{ for all } (x, y) \in \mathbb{H} \quad (3.1) \]

\[ \int_{\mathbb{R}} P(x, y) dx = 1 \text{ for all } y > 0 \quad (3.2) \]

\[ \lim_{y \to 0} \int_{|x|>\epsilon} P(x, y) dx = 0 \text{ for all } \epsilon > 0 \quad (3.3) \]

\[ P(x, y) \text{ is harmonic in } \mathbb{H} \quad (3.4) \]

Proof. The property 3.1 is evident from the definition. For 3.2 and 3.3 we do a simple calculation:

\[ \int_{a}^{b} P(x, y) dx = \frac{1}{\pi} \int_{a}^{b} \frac{y}{x^2 + y^2} dx = \frac{1}{\pi} \int_{a}^{b} \frac{1/y}{(x/y)^2 + 1} dx = \]

\[ \frac{1}{\pi} (\arctan(b/y) - \arctan(a/y)). \]

We can use this to prove 3.2 by letting \( b \to \infty \) and \( a \to -\infty \). Then we may prove 3.3 by letting \( a = -\epsilon \) and \( b = \epsilon \), calculating that

\[ \int_{|x|>\epsilon} P(x, y) dx = 1 - \int_{-\epsilon}^{\epsilon} P(x, y) dx = 1 - \frac{2}{\pi} \arctan(\epsilon/y). \]

This proves 3.3 since the right hand side approaches zero as \( y \to 0 \).

For 3.4 we notice that \( f(z) = i/\pi z \) is analytic in \( \mathbb{H} \). Thus \( \text{Re } f \) is harmonic. But \( \text{Re } f = \text{Re } i/\pi(x+iy) = \text{Re } i(x-iy)/\pi(x^2 + y^2) = y/\pi(x^2 + y^2) = P(x, y) \) and thus \( P \) is harmonic. ☐

Now we are ready to solve our special case of the Dirichlet problem, stated as the following theorem:

Theorem 3.2. Let \( f : \mathbb{R} \mapsto \mathbb{R} \) be a continuous and bounded function. Then there exists a harmonic function \( u : \mathbb{H} \mapsto \mathbb{R} \) such that \( \lim_{z \to x} u(z) = f(x) \) for all \( x \in \mathbb{R} \). More specifically it is given by

\[ u(x, y) = \int_{\mathbb{R}} f(t) P(x - t, y) dt = \frac{1}{\pi} \int_{\mathbb{R}} f(t) \frac{y}{(x-t)^2 + y^2} dt. \]
Proof. We will first make the remark that $u$ is well-defined since $f$ is bounded and $P(x - t, y)$ is integrable for any fixed $y$. It is also harmonic, since $f(t)P(x - t, y)$ and its partial derivatives with respect to $x$ and $y$ are continuous in $\mathbb{H}$ and thus we may differentiate under the integral sign to see that

$$\Delta u(x, y) = \Delta \int_{\mathbb{R}} f(t)P(x - t, y)dt = \int_{\mathbb{R}} f(t)\Delta P(x - t, y)dt = \int_{\mathbb{R}} 0 dt = 0,$$

where we have used the harmonicity of the Poisson kernel. It remains to prove that

$$\lim_{z \to x_0} u(z) = f(x_0)$$

for any fixed $x_0 \in \mathbb{R}$. Let $\epsilon > 0$ and write $z = x + iy$. Note that $z \to x_0$ implies that $x \to x_0$ and $y \to 0$. Let us now use property 3.2 in Theorem 3.1 to compute that

$$|u(x, y) - f(x_0)| = |\int_{\mathbb{R}} f(t)P(x - t, y)dt - \int_{\mathbb{R}} f(x_0)P(x - t, y)dt|$$

$$= |\int_{\mathbb{R}} (f(t) - f(x_0))P(x - t, y)dt|$$

$$\leq \int_{\mathbb{R}} |f(t) - f(x_0)|P(x - t, y)dt$$

$$= \int_{|x-t|<\delta'} |f(t) - f(x_0)|P(x - t, y)dt$$

$$+ \int_{|x-t|>\delta'} |f(t) - f(x_0)|P(x - t, y)dt$$

for all $\delta' > 0$. We will now show that for $z$ close enough to $x_0$ and appropriately chosen $\delta'$ the two integrals in the last expression are both less than $\epsilon/2$. Since $x$ is close to $x_0$, by the continuity of $f$ we may assume that $|f(x) - f(x_0)| < \epsilon/4$. Choose $\delta'$ small enough so that $|f(t) - f(x)| < \epsilon/4$ when $|x - t| < \delta'$. These imply that $|f(t) - f(x_0)| < \epsilon/2$ when $|x - t| < \delta'$, after which we have the estimate

$$\int_{|x-t|<\delta'} |f(t) - f(x_0)|P(x - t, y)dt \leq \frac{\epsilon}{2} \int_{|x-t|<\delta'} P(x - t, y)dt \leq \frac{\epsilon}{2} ||P||_1 = \frac{\epsilon}{2}.$$

By the boundedness of $f$ there exists a $K$ such that $|f(x)| < K$ for all $x \in \mathbb{R}$. Now we see that

$$\int_{|x-t|>\delta'} |f(t) - f(x)|P(x - t, y)dt \leq 2K \int_{|x-t|>\delta'} P(x - t, y)dt.$$

By property 3.3 in Theorem 3.1, this is less than $\epsilon/2$ for sufficiently small $y$. Thus

$$|u(x, y) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $z$ sufficiently close to $x_0$, which proves the theorem. \qed
3.2 Conjugate functions

Let again \( f : \mathbb{R} \to \mathbb{R} \) be continuous and bounded. Theorem 3.2 gives us the harmonic function \( u : \mathbb{H} \to \mathbb{R} \) with \( f \) as its boundary limit. Since \( \mathbb{H} \) is a convex domain, there must exist another harmonic function \( v : \mathbb{H} \to \mathbb{R} \) such that \( u + iv \) is analytic in \( \mathbb{H} \). We also see that \( v \) is unique up to a constant, since if both \( u + iv_1 \) and \( u + iv_2 \) are analytic in \( \mathbb{H} \), then so must be \((u + iv_1) - (u + iv_2) = i(v_1 - v_2)\). The Cauchy-Riemann equations then imply that \( v_1 - v_2 \) is constant as wanted. It is also relevant to ask if we could characterize the solutions for \( v \) in terms of \( f \). This is indeed possible, but we will have to tighten the assumptions on \( f \). Let us start with a definition.

**Definition 3.3.** We define the conjugate Poisson kernel \( Q(x, y) \) by

\[
Q(x, y) = \frac{1}{\pi} \frac{x}{x^2 + y^2}
\]

for all \((x, y) \in \mathbb{H}\).

The name is appropriate because \( P + iQ = i/(\pi z) \) is analytic in \( \mathbb{H} \), and thus \( Q \) is a conjugate function of \( P \). This also shows that \( Q \) is harmonic in \( \mathbb{H} \). But in some sense the conjugate Poisson kernel is more of an evil twin of the Poisson kernel. We would like to be able to define

\[
v(x, y) = \int_{\mathbb{R}} f(t)Q(x - t, y)dt
\]

for all \((x, y) \in \mathbb{H}\), but we run into complications since \( Q(x, y) \) decays slower than \( P(x, y) \), which makes the integral undefined for a general continuous and bounded function \( f \). If we however assume that \( f \) is also integrable over \( \mathbb{R} \), then the previous integral is well-defined since \( Q(x, y) \) is bounded for any fixed \( y > 0 \). It also defines a conjugate function of \( u \) since \( P \) and \( Q \) are conjugate functions, and applying differentiation under the integral sign and the Cauchy-Riemann equations for \( P \) and \( Q \) shows that \( u \) and \( v \) satisfy them too.

We have now defined \( v \), one of the conjugate functions of \( u \), with a formula that is quite natural. There is however one more interesting question remaining in this setting: What are the boundary values of \( v \)? It turns out that if \( x_0 \in \mathbb{R} \), then the limit

\[
\lim_{z \to x_0} v(z)
\]

is the Hilbert transform of \( f \) at point \( x_0 \). Formally we have that

\[
\lim_{x+iy \to x_0} \int_{\mathbb{R}} \frac{(x-t)f(t)}{\pi((x-t)^2 + y^2)} dt = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x_0-t} dt = \mathcal{H}f(x_0).
\]
However, we didn’t even assume that $Hf$ exists. But if we tighten the assumptions on $f$ once again this limit turns out to be true rigorously, which we will state as the following theorem.

**Theorem 3.3.** Assume $f : \mathbb{R} \to \mathbb{R}$ such that $f$ is continuously differentiable and has compact support. Define $v(x, y)$ as in (3.5). Then

$$\lim_{z \to x_0} v(z) = Hf(x_0)$$

for all $x_0 \in \mathbb{R}$.

**Proof.** The assumptions on $f$ imply that the Hilbert transform $Hf$ exists as discussed in the introductory chapter. We may also calculate it as the pointwise limit of $H_\epsilon f$ as $\epsilon \to 0$. Hence it will be enough to prove that $|v(x, \epsilon) - H_\epsilon(x_0)|$ is arbitrarily small for $x$ close enough to $x_0$ and $\epsilon$ small enough. For this we first note that

$$|v(x, \epsilon) - H_\epsilon(x_0)| \leq |v(x, \epsilon) - H_\epsilon(x)| + |H_\epsilon(x) - H_\epsilon(x_0)|. \quad (3.6)$$

We will now estimate the two terms on the right hand side. For the second one we see that

$$\pi |H_\epsilon(x) - H_\epsilon(x_0)| \leq \int_{|t| \geq 1} \frac{|f(x-t) - f(x_0-t)|}{t} dt + \int_{1>|t|>\epsilon} \frac{|f(x-t) - f(x_0-t)|}{t} dt \leq \int_{|t| \geq 1} |f(x-t) - f(x_0-t)| dt + \int_{1>|t|} \left| \frac{(f(x-t) - f(x)) - (f(x_0-t) - f(x_0))}{t} \right| dt.$$

Now we may apply the dominated convergence theorem on the terms on the utmost right hand side. The first term converges to zero as $x \to x_0$ because $f$ has compact support. The second one does the same since the expression

$$\left| \frac{(f(x-t) - f(x)) - (f(x_0-t) - f(x_0))}{t} \right|$$

is bounded on $[-1, 1]$ by the mean value theorem and converges to zero pointwise.
For the other term in (3.6) we estimate that
\[
\pi |v(x, \epsilon) - \mathcal{H}_\epsilon f(x)| = \int_{\mathbb{R}} \frac{tf(x-t)}{t^2 + \epsilon^2} dt - \int_{|t|>\epsilon} \frac{f(x-t)}{t} dt
\]
\[
= \int_{\mathbb{R}} \frac{tf(x-\epsilon t)}{t^2 + 1} dt - \int_{|t|>\epsilon} \frac{f(x-\epsilon t)}{t} dt
\]
\[
= \int_{|t|\leq 1} \frac{tf(x-\epsilon t)}{t^2 + 1} dt + \int_{|t|>1} \left( \frac{t}{t^2 + 1} - \frac{1}{t} \right) f(x-\epsilon t) dt
\]
\[
= \int_{|t|\leq 1} \frac{tf(x-\epsilon t)}{t^2 + 1} dt + \int_{|t|>1} \frac{-f(x-\epsilon t)}{t(t^2 + 1)} dt.
\]

We now estimate both of the integrands on the utmost right hand side. Applying the boundedness of \(f\), we get that
\[
\left| \frac{tf(x-\epsilon t)}{t^2 + 1} \right| \leq K|t| \quad \text{and} \quad \left| \frac{f(x-\epsilon t)}{t(t^2 + 1)} \right| \leq \frac{K}{|t||(t^2 + 1)}
\]
for some constant \(K > 0\). These are integrable in their respective sets, and thus the dominated convergence theorem implies that
\[
\lim_{\epsilon \to 0} \left[ \int_{|t|\leq 1} \frac{tf(x-\epsilon t)}{t^2 + 1} dt + \int_{|t|>1} \frac{-f(x-\epsilon t)}{t(t^2 + 1)} dt \right]
\]
\[
= \int_{|t|\leq 1} \frac{tf(x)}{t^2 + 1} dt + \int_{|t|>1} \frac{-f(x)}{t(t^2 + 1)} dt
\]
\[
= 0,
\]
as both of the kernels \(t/(t^2 + 1)\) and \(1/t(t^2 + 1)\) are odd. This finishes the proof. \(\Box\)
4 Continuous linear extensions

In preparation for proving the boundedness of the Hilbert transform we will now revisit the so-called continuous linear extension theorem, which gives means of extending continuous linear transformations. The strategy we will use in proving the boundedness of the Hilbert transform is not to show that our given definition of it is valid for all $L^p$-functions for $1 \leq p < \infty$, but that it satisfies the wanted bounds for the functions we already know it is well-defined for (for example $C_0^\infty$-functions), and extend it to the whole class of $L^p$-functions using density arguments and other sneaky means. The continuous linear extension theorem proves to be especially useful for this. We will prove a slight generalization of the classical version to suit our needs, and for this we will first introduce some basic facts about quasinorms.

**Definition 4.1.** Let $X$ be a vector space. A function $|| \cdot || : X \mapsto [0, \infty)$ is a *quasinorm*, if for all scalars $a$ and $x \in X$ we have that

$$||ax|| = |a|||x||$$

and

$$||x|| = 0 \iff x = \bar{0}. \tag{4.2}$$

In addition, we require that there exists a constant $K > 0$ such that

$$||x + y|| \leq K(||x|| + ||y||) \tag{4.3}$$

for all $x, y \in X$. We say that the pair $(X, || \cdot ||)$ is a *quasinormed space*.

**Remark 4.2.** Every norm is a quasinorm, since equation (4.3) becomes equivalent with the triangle inequality if we set $K = 1$. In quasinormed spaces we define concepts such as converging sequences, continuous functions, etc. analogously to the definitions in normed spaces, and it happens that most of the properties stay the same. A particular result is that a linear transformation $T$ from a quasinormed space $(X, || \cdot ||_X)$ to another quasinormed space $(Y, || \cdot ||_Y)$ is continuous if and only if its norm $||T||$ is finite, which is proved in the same way as its counterpart in normed spaces. Here the operator norm is defined in the usual way as

$$||T|| = \sup\{||Tx||_Y : x \in X \text{ with } ||x||_X = 1\}.$$

**Definition 4.3.** A quasinormed space $(X, || \cdot ||)$ is a *quasi-Banach space* if every sequence in $X$ that is Cauchy with respect to $|| \cdot ||$ has a limit in $X$. 

11
Here a sequence \((x_n)\) is Cauchy in \(X\) if for every \(\epsilon > 0\) there exists \(n_\epsilon \in \mathbb{Z}_+\) such that
\[
||x_p - x_q|| < \epsilon
\]
for all \(p, q \in \mathbb{Z}_+\) such that \(p, q \geq n_\epsilon\).

**Theorem 4.1. (Continuous linear extension theorem).** Let \((X, ||\cdot||_X)\) be a normed vector space and \(M\) be a dense vector subspace of \(X\). Let also \((Y, ||\cdot||_Y)\) be a quasi-Banach space, and \(T : M \mapsto Y\) be a continuous linear transformation. Then there exists a unique continuous linear transformation \(\tilde{T} : X \mapsto Y\) such that \(\tilde{T}|_M = T\). In addition, if the following condition for the space \(Y\) holds

\[
\text{If } (x_n) \subset Y \text{ is a sequence converging to some } x \in Y \text{ with respect to } ||\cdot||_Y, \text{ then } ||x_n||_Y \to ||x||_Y \text{ as } n \to \infty. \tag{4.4}
\]
then we have that \(||\tilde{T}|| = ||T||\).

**Proof.** Assume that \(z \in X\). Then since \(M\) is dense in \(X\) there exists a sequence \((x_n) \subset M\) such that \(x_n \to z\) in \(X\). Especially it follows that the sequence \((x_n)\) is Cauchy in \(X\). Then if \(p, q \in \mathbb{Z}_+\), by the continuity of \(T\) we have that
\[
||Tx_p - Tx_q||_Y = ||T(x_p - x_q)||_Y \leq ||T|| ||x_p - x_q||_X,
\]
and we easily see that \((T(x_n))\) is a Cauchy sequence in \(Y\) and by completeness it converges to a limit we will call \(\tilde{T}(z)\). We will now prove that \(\tilde{T}(z)\) only depends on \(z\) and not on the sequence \((x_n)\), and thus that \(\tilde{T}\) is a well-defined function from \(X\) to \(Y\). For this assume that \((x_n) \subset M\) and \((y_n) \subset M\) are two sequences converging to some \(z \in X\). We make the following remark:

If \((a_n)\) and \((b_n)\) are two sequences in \(Y\) converging to \(a\) and \(b\) respectively and \(\alpha, \beta\) are real numbers, then the sequence \((\alpha a_n + \beta b_n)\) converges and its limit is \(\alpha a + \beta b\).

This is true because \(||\cdot||_Y\) is a quasinorm and thus there exists a constant \(K\) such that \(||x + y||_Y \leq K(||x||_Y + ||y||_Y)\) for all \(x, y \in Y\). We then get the estimate
\[
||\alpha a_n + \beta b_n - \alpha a - \beta b||_Y \leq K||\alpha|| ||a_n - a||_Y + K||\beta|| ||b_n - b||_Y
\]
which proves the remark since the right hand side approaches zero as \(n \to \infty\). Returning back to what we wanted to prove, we have that
\[
\lim_{n \to \infty} ||x_n - y_n||_X = ||z - z||_X = 0.
\]
Now the continuity of $T$ and (4.5) imply that
\[ 0 = \lim_{n \to \infty} T(x_n - y_n) = \lim_{n \to \infty} (T(x_n) - T(y_n)) = \lim_{n \to \infty} T(x_n) - \lim_{n \to \infty} T(y_n). \]
This is justified because both of the sequences $(T(x_n))$ and $(T(y_n))$ have limits by completeness, and we conclude that they must be the same, which is what we wanted to prove. By the continuity of $T$ we also get that $\tilde{T}(z) = T(z)$ for $z \in M$.

We will now show that $\tilde{T}$ is linear. For this assume that $\alpha$ and $\beta$ are real numbers and that $x, y \in X$. We choose two sequences, $(x_n)$ and $(y_n)$, converging to $x$ and $y$ respectively. Then (4.5) implies that
\[
\tilde{T}(\alpha x + \beta y) = \lim_{n \to \infty} T(\alpha x_n + \beta y_n) = \lim_{n \to \infty} (\alpha T(x_n) + \beta T(y_n))
= \alpha \lim_{n \to \infty} T(x_n) + \beta \lim_{n \to \infty} T(y_n) = \alpha \tilde{T}(x) + \beta \tilde{T}(y)
\]
and thus $\tilde{T}$ is linear.

For the continuity of $\tilde{T}$ it then suffices to prove that $\tilde{T}$ is a bounded linear transformation. For this fix $z \in X$ with $||z||_X = 1$. We may also choose a sequence $T(x_n) \in M$ converging to $z$ with $||x_n||_X = 1$ for all $n$. Since $T(x_n)$ converges to $\tilde{T}(z)$, we have that $||\tilde{T}(z) - T(x_{n_0})||_Y \leq 1$ for some $n_0$. Thus
\[
||\tilde{T}(z)||_Y \leq K(||\tilde{T}(z) - T(x_{n_0})||_Y + ||T(x_{n_0})||_Y)
\leq K(1 + ||T||_X||x_{n_0}||_X) = K(1 + ||T||),
\]
which proves that $\tilde{T}$ is bounded. If however condition (4.4) holds, by the assumption in the theorem we have that $||T(x_n)||_Y$ converges to $||T(z)||_Y$, and thus $||\tilde{T}|| \leq ||T||$, which proves that $||T|| = ||T||$ since $\tilde{T}$ agrees with $T$ on $M$.

It remains to prove that $\tilde{T}$ is unique. For this assume that $S : X \mapsto Y$ is any continuous linear transformation such that $S|_M = T$. Then $S - \tilde{T}$ is a continuous linear transformation that vanishes in $M$. But since $M$ is dense in $X$, by continuity $S - \tilde{T}$ must also vanish in $X$, and thus $S = \tilde{T}$. \qed
5 $L^1$ theory of the Hilbert transform

It turns out that the behavior of the Hilbert transform on $L^1$ is somehow worse than on the other spaces $L^p$ for $1 < p < \infty$. The problem is that $H$ doesn’t map $L^1$ into $L^1$, which can be verified by considering the characteristic function of the interval $[0, 1]$. Calculating the Hilbert transform by the given definition yields that $H\chi_{[0,1]}(x) = 1/\pi \log |x/(x-1)|$ which obviously isn’t integrable on $\mathbb{R}$. But it also turns out that this problem can be averted by replacing $L^1$ with a larger codomain. This will be the so-called weak-$L^1$ space.

**Definition 5.1.** Let $1 \leq p < \infty$. The weak-$L^p$ space, denoted by $L^{p,w}$, is defined to be the set of (equivalence classes of) measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ for which there exists a constant $C$ such that

$$|\{x \in \mathbb{R} : |f(x)| > t\}| \leq \frac{C}{t^p}$$

for all $t > 0$. The smallest constant $C$ for which this holds is called the norm of $f$ and it is denoted by $||f||_{p,w}$. For measurable functions $f$ not in $L^{p,w}$ we say that $||f||_{p,w} = \infty$.

We also define the weak-$L^\infty$ space to be the same space as $L^\infty$.

**Remark 5.2.** If $f \in L^p$ for some $p$ with $1 \leq p < \infty$, we may apply Markov’s inequality on the integrable function $f^p$ to deduce that

$$|\{x \in \mathbb{R} : |f(x)| > t\}| \leq \frac{||f||_{p,w}^p}{t^p}.$$  

This implies that $L^p \subseteq L^{p,w}$ and that $||f||_{p,w} \leq ||f||_p$.

In this section our main aim is to extend the Hilbert transform to a bounded linear transformation from $L^1$ to $L^{1,w}$. We must first make a remark that the $L^{1,w}$-norm is not a real norm, but a quasinorm. But this won’t be much of a problem as our continuous linear extension theorem (Theorem 4.1) also includes quasinormed spaces. Thus we will first verify that the conditions in Theorem 4.1 hold.

**Theorem 5.1.** The pair $(L^{1,w}, ||\cdot||_{1,w})$ is a quasi-Banach space. In addition, it satisfies condition (4.4) in the continuous linear extension theorem.

**Proof.** First we note that if we have two measurable functions $f$ and $g$, and $a, b > 0$ with $a + b = 1$, then

$$|\{|f + g| > t\}| \leq |\{|f| > at\}| + |\{|g| > bt\}| \leq \frac{||f||_{1,w}}{at} + \frac{||g||_{1,w}}{bt}.$$
for all \( t > 0 \) and thus \( \|f + g\|_{1,w} \leq \|f\|_{1,w}/a + \|g\|_{1,w}/b \). This already tells us that \( f + g \in L^{1,w} \) for all \( f, g \in L^{1,w} \), and now it is easy to verify that \( L^{1,w} \) is a vector space. Plugging in \( a = b = 1/2 \) into the previous reasoning yields that \( \|f + g\|_{1,w} \leq 2(\|f\|_{1,w} + \|g\|_{1,w}) \) and thus \( \|\cdot\|_{1,w} \) is indeed a quasinorm (verifying the other axioms of a quasinorm is trivial). If we assume that \( f, g \in L^{1,w} \) and instead put in \( a = \sqrt{\|f\|_{1,w}} \sqrt{\|f\|_{1,w}} + \sqrt{\|g\|_{1,w}} \) and \( b = \sqrt{\|g\|_{1,w}} \sqrt{\|f\|_{1,w}} + \sqrt{\|g\|_{1,w}} \) we get after simplifying that \( \|f + g\|_{1,w} \leq \sqrt{\|f\|_{1,w} + \sqrt{\|f\|_{1,w}}^2} \) and thus

\[
\sqrt{\|f + g\|_{1,w}} \leq \sqrt{\|f\|_{1,w} + \sqrt{\|g\|_{1,w}}}. \tag{5.1}
\]

This result proves to be useful since if \( (f_n) \subset L^{1,w} \) is a sequence of functions converging to another measurable function \( f \) with respect to the quasinorm, then \( f \) must also be in \( L^{1,w} \) and by (5.1) we get that

\[
\left| \sqrt{\|f\|_{1,w}} - \sqrt{\|f_n\|_{1,w}} \right| \leq \sqrt{\|f - f_n\|_{1,w}}.
\]

And since the above implies that \( \|f_n\|_{1,w} \to \|f\|_{1,w} \) as \( n \to \infty \), we have proven the additional condition (4.4) in the continuous linear extension theorem.

It remains to prove that \( L^{1,w} \) is complete. We will imitate the proof of the Riesz-Fischer theorem in proving this. Let us first prove an analogue of Fatou’s lemma. If \( f_1, f_2, \ldots \) is a sequence of nonnegative measurable functions in \( \mathbb{R} \), then we claim that

\[
|\{ x \in \mathbb{R} : \liminf_{n \to \infty} f_n(x) > t \}| \leq \liminf_{n \to \infty} |\{ x \in \mathbb{R} : f_n(x) > t \}| \tag{5.2}
\]

for all \( t > 0 \). The proof is also quite analogous to the proof of Fatou’s lemma. Define nonnegative functions \( g_k \) by

\[
g_k(x) = \inf_{k \leq n} f_n(x).
\]

We have that \( (g_k) \) is an increasing sequence of functions, and thus

\[
\{ x : g_k(x) > t \} \subset \{ x : g_{k+1}(x) > t \}
\]

15
for all \( k \). This implies that
\[
\{ x : \lim_{n \to \infty} g_n(x) > t \} = \lim_{n \to \infty} | \{ x : g_n(x) > t \} | = \liminf_{n \to \infty} | \{ x : g_n(x) > t \} |
\]
and therefore
\[
\{ x : \liminf_{n \to \infty} f_n(x) > t \} = \liminf_{n \to \infty} | \{ x : g_n(x) > t \} |
\]
\[
= \liminf_{n \to \infty} | \{ x : f_n(x) > t \} |,
\]
where the last inequality is due to the fact that \( g_n \leq f_n \) for all \( n \). This proves (5.2).

We will now assume that \( f_1, f_2, \ldots \) is a Cauchy sequence in \( L^{1,w} \). First we will extract a subsequence converging pointwise almost everywhere. Choose positive integers \( n_1 < n_2 < n_3 < \ldots \) in such a way that
\[
||f_{n_{k+1}} - f_{n_k}||_{1,w} < 4^{-k}
\]
for all \( k \). Let us define for \( m = 1, 2, \ldots \) the measurable functions
\[
g_m(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)|.
\]
Since the sequence \((g_m)\) is increasing, it must converge to some measurable function \( g \). We first show that \( g(x) < \infty \) for almost every \( x \). For this apply equation (5.1) to deduce that
\[
\sqrt{||g_m||_{1,w}} \leq \sqrt{||f_{n_1}||_{1,w}} + \sum_{k=1}^{m} \sqrt{||f_{n_{k+1}} - f_{n_k}||_{1,w}}
\]
\[
< \sqrt{||f_{n_1}||_{1,w}} + \sum_{k=1}^{\infty} \frac{1}{4^k} = \sqrt{||f_{n_1}||_{1,w}} + 1.
\]
This also shows that \( ||g||_{1,w} \leq (\sqrt{||f_{n_1}||_{1,w}+1})^2 \), which implies that \( g(x) < \infty \) for almost every \( x \) as wanted. Thus the series
\[
|f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|
\]
converges for almost every \( x \), which implies the same for the series
\[
f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)).
\]
But the $m$:th partial sum of this series is just $f_{nm}(x)$, which means that there exists a measurable function $f$ such that

$$f_{nk} \to f$$

pointwise almost everywhere as $k \to \infty$.

We still have to prove that $f$ is in $L^{1,w}$ and that it is the $L^{1,w}$-limit of $f_n$ as $n \to \infty$. For this let $\epsilon > 0$, and choose $j_0$ such that

$$||f_n - f_m||_{1,w} < \epsilon$$

for all $n, m \geq m_0$. If $n \geq m_0$ then (5.2) gives us the estimate

$$|\{x : |f_n(x) - f(x)| > t\}| = |\{x : \lim_{k \to \infty} |f_n(x) - f_{nk}(x)| > t\}|$$

$$\leq \liminf_{k \to \infty} |\{x : |f_n(x) - f_{nk}(x)| > t\}|$$

$$\leq \liminf_{k \to \infty} \frac{||f_n - f_{nk}||_{1,w}}{t} \leq \frac{\epsilon}{t}$$

for all $t > 0$, which proves that $||f_n - f||_{1,w} < \epsilon$. Thus $f = f_n - (f - f_n)$ is in $L^{1,w}$ and $f_n \to f$ in $L^{1,w}$ as wanted. This concludes the proof of Theorem 5.1.

Thus the conditions imposed on the domain and codomain in the continuous linear extension theorem are fulfilled. For our proof of the weak boundedness of the Hilbert transform it remains to choose a dense vector subspace $M$ of $L^1$ in which the Hilbert transform is well-defined and on which it is bounded with respect to the $L^{1,w}$-norm and deduce that it has a unique continuous linear extension to the whole $L^1$. In general we say that an operator $T$ from $L^p$ to weak-$L^p$ is weak $(p, p)$ if

$$||Tf||_{p,w} \leq C||f||_p$$

(5.3)

for all $f \in L^p$. In the case of the Hilbert transform it remains to prove that it is weak $(1, 1)$. This can be reduced to finding a constant $C$ for which

$$|\{x : |Hf(x)| > t\}| \leq \frac{C||f||_1}{t}$$

for all $t > 0$. We will also call this the weak $(1, 1)$ inequality. Proving that the Hilbert transform is weak $(1, 1)$ is usually done by applying the so-called Calderón-Zygmund decomposition of a function $f$, but for the sake of adventure we will do it in another way. We will instead replicate an original proof by Guzman, see [2], but will present a shorter although less general version of it. The idea is the following:
Show that the Hilbert transform satisfies the weak \((1, 1)\) inequality for finite linear combinations of Dirac delta functions, where the Hilbert transform of the Dirac delta function \(\delta(x)\) is defined to be \(1/\pi x\).

Use this to extend the same fact to the truncated Hilbert transforms.

Approximate a general nonnegative \(L^1\) function of compact support by Dirac deltas and reason that the truncated Hilbert transforms satisfy the weak \((1, 1)\) inequality.

Use this to prove that the Hilbert transform satisfies the weak \((1, 1)\) inequality for smooth functions of compact support.

After this we are done, since smooth functions of compact support are dense in \(L^1\) and they form a vector subspace. We will collect the first two points into lemmas:

**Lemma 5.2.** Let \(c_1, \ldots, c_n\) be nonnegative real numbers and \(a_1 < \cdots < a_n\) any real numbers. We define the measure \(f(x) = \sum_{k=1}^{n} c_k \delta(x - a_k)\). Then for any \(t > 0\) the following equality is valid:

\[
|\{x \in \mathbb{R} : |\mathcal{H} f(x)| > t\}| = |\{x \in \mathbb{R} : \frac{1}{\pi} \sum_{k=1}^{n} \frac{c_k}{x - a_k} > t\}| = \frac{2}{\pi t} \sum_{k=1}^{n} c_k.
\]

**Proof.** The first equality is clear since by definition,

\[
\mathcal{H} f(x) = \frac{1}{\pi} \sum_{k=1}^{n} \frac{c_k}{x - a_k}.
\]

Let us first consider the behavior of \(\mathcal{H} f\) on the real axis. It is clear that \(\mathcal{H} f\) is a continuous function on the intervals \((a_k, a_{k+1})\) for \(k = 1, \ldots, n - 1\) and on two unbounded intervals \((-\infty, a_1)\) and \((a_n, \infty)\). Due to the singular behavior of the terms \(c_k/(x - a_k)\) in \(\mathcal{H} f\), \(\lim_{x \to a_k^+} \mathcal{H} f(x) = \infty\) and \(\lim_{x \to a_k^-} \mathcal{H} f(x) = -\infty\) for all \(k\). We also see that \(\mathcal{H} f(x)\) approaches zero from below as \(x \to -\infty\) and from above as \(x \to \infty\). Thus the equation \(\mathcal{H} f(x) = \lambda\) for a fixed \(\lambda \neq 0\) has at least \(n\) real roots. On the other hand, this equation is equivalent with

\[
\sum_{k=1}^{n} c_k (x - a_1) \cdots (x - a_{k-1})(x - a_{k+1}) \cdots (x - a_n) = \pi \lambda (x - a_1) \cdots (x - a_n).
\]

\(^1\)It should be noted that this lemma is a straightforward generalization of problem 1988/4 of the International Mathematical Olympiad.
Which is just a polynomial equation of degree $n$ and thus has at most $n$ roots. From this we conclude that the equation has exactly $n$ roots.

Let us now fix a $t > 0$. From the above we conclude that if the solutions of the equation $Hf(x) = t$ are $x_1, \ldots, x_n$ in increasing order, then the solution set of the inequality $Hf(x) > t$ is the union of the intervals $(a_1, x_1), \ldots, (a_n, x_n)$ and thus has measure $\sum(x_k - a_k)$. Since $x_1, \ldots, x_n$ are the roots of the polynomial

$$\pi t(x - a_1) \cdots (x - a_n) - \sum_{k=1}^n c_k (x - a_1) \cdots (x - a_{k-1})(x - a_{k+1}) \cdots (x - a_n),$$

by Vieta's relations we get that

$$\sum_{k=1}^n x_k = -\frac{\pi t \sum_{k=1}^n -a_k - \sum_{k=1}^n c_k}{\pi t} = \frac{\sum_{k=1}^n c_k}{\pi t} + \sum_{k=1}^n a_k.$$

And it follows that

$$|\{x \in \mathbb{R} : Hf(x) > t\}| = \sum_{k=1}^n (x_k - a_k) = \sum_{k=1}^n x_k - \sum_{k=1}^n a_k = \frac{\sum_{k=1}^n c_k}{t}.$$

By a similar computation we see that $|\{H f(x) < -t\}| = (\sum_{k=1}^n c_k)/\pi t$ and combining these two equalities yields the desired equality. \qed

**Lemma 5.3.** Let $c_1, \ldots, c_n$ be nonnegative real numbers and $a_1 < \cdots < a_n$ any real numbers and let $f$ be as in Lemma 5.2. Let also $\epsilon > 0$. Then there exists a constant $C$ independent of $f$ and $\epsilon$ for which

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| = |\{x \in \mathbb{R} : |\frac{1}{\pi} \sum_{k=1}^n c_k X_{\epsilon}(x-a_k)| > t\}| \leq \frac{C}{\pi t} \sum_{k=1}^n c_k$$

**Proof.** Choose a compact subset $K$ of the set $\{Hf(x) > t\} \setminus \{a_1, \ldots, a_n\}$. We may now choose real numbers $x_1, \ldots, x_m \in K$ in such a way that the intervals $I_k = [x_k - \epsilon, x_k + \epsilon]$ are disjoint and that $|K| \leq 2|\bigcup_{k=1}^m I_k|$. This is justified because if $|K| > 2|\bigcup_{k=1}^m I_k|$ for some already chosen $I_k$ then $K \setminus \bigcup_{k=1}^m [x_k - 2\epsilon, x_k + 2\epsilon]$ must be nonempty and contains an element we will call $x_{k+1}$. It is easy to verify that $I_{k+1} = [x_{k+1} - \epsilon, x_{k+1} + \epsilon]$ and the $I_k$ are disjoint so we have increased our collection of intervals by one. The compactness of $K$ yields that this process stops at some point.
We now define for $k = 1, \ldots, m$ the measures $f_k$ and $f_k^*$ by setting
\[
f_k(x) = f(x)\chi_{I_k}(x) = \frac{1}{\pi} \sum_{a_k \in I_k} c_k \delta(x - a_k)
\]
and
\[
f_k^*(x) = f(x)\chi_{\mathbb{R} \setminus I_k}(x) = \frac{1}{\pi} \sum_{a_k \notin I_k} c_k \delta(x - a_k).
\]
Thus $f = f_k + f_k^*$ for all $k$. Furthermore, $\mathcal{H}f(x_k) = \mathcal{H}(f\chi_{\mathbb{R} \setminus I_k})(x_k) = \mathcal{H}f_k^*(x_k)$ and therefore $|\mathcal{H}f_k^*(x_k)| > t$. Now we see that
\[
\mathcal{H}f_k^*(x) = \frac{1}{\pi} \sum_{a_k \notin I_k} \frac{c_k}{x - a_k}.
\]
This sum doesn’t contain any singularities on the interval $I_k$, and thus by the decreasing nature of the function $1/x$ it must be decreasing on $I_k$. Since $|\mathcal{H}f_k^*(x_k)| > t$, it must be that $|\mathcal{H}f_k^*(x)| > t$ on at least one of the intervals $[x_k, x_k + \epsilon]$ and $[x_k - \epsilon, x_k]$. For each $k$ let this interval be denoted by $I_k^*$, obviously satisfying $|I_k| = 2|I_k^*|$. It follows that $I_k^* \subset \{ |\mathcal{H}f_k^*| > t \}$ and therefore
\[
|K| \leq 2 \bigcup_{k=1}^m I_k = 4 \bigcup_{k=1}^m I_k^* \leq 4 \bigcup_{k=1}^m \{ |\mathcal{H}f_k^*| > t \}.
\]
Then we see that $|\mathcal{H}f_k^*| = |\mathcal{H}f - \mathcal{H}f_k| \leq |\mathcal{H}f| + |\mathcal{H}f_k|$ and thus
\[
\{ |\mathcal{H}f_k^*| > t \} \subset \{ |\mathcal{H}f| > t/2 \} \cup \{ |\mathcal{H}f_k| > t/2 \}.
\]
Hence
\[
\bigcup_{k=1}^m \{ |\mathcal{H}f_k^*| > t \} \subset \{ |\mathcal{H}f| > t/2 \} \cup \bigcup_{k=1}^m \{ |\mathcal{H}f_k| > t/2 \}.
\]
Finally Lemma 5.2 yields the estimate
\[
|K| \leq 4 \left( \frac{4}{\pi t} ||f||_1 + \frac{4}{\pi t} \sum_{k=1}^m ||f_k||_1 \right) \leq \frac{32}{\pi t} ||f||_1.
\]
Since $K$ was an arbitrary, $|K|$ gets as close to $\{ |\mathcal{H}f| > t \}$ as we want and the lemma is proven.

We will now apply this lemma to deduce the following theorem.

20
Theorem 5.4. Let \( g \in L^1 \) be nonnegative and have compact support. Then the truncated Hilbert transforms \( \mathcal{H}_\epsilon \) of \( g \) satisfy the weak \((1,1)\) inequality, that is
\[
|\{ x \in \mathbb{R} : |\mathcal{H}_\epsilon g(x)| > t \}| \leq \frac{C||g||_1}{t}
\]
for some constant \( C \) independent of \( g \) and \( \epsilon \).

Proof. Let us divide the support of \( g \) into \( n \) intervals \( I_1, \ldots, I_n \). Define
\[
f(x) = \sum_{k=1}^{n} \| g \cdot \chi_{I_k} \|_1 \delta(x - a_k)
\]
where \( a_k \) is the left endpoint of the interval \( I_k \). Note that since \( n \) was arbitrary, we may at any point assume that the intervals \( I_k \) are as small as we please. Note also that \( ||f||_1 = ||g||_1 \). Now fix \( t \) and for \( t > \alpha > 0 \), we make the estimate
\[
|\{ |\mathcal{H}_\epsilon g| > t \}| \leq |\{ |\mathcal{H}_\epsilon f| > t - \alpha \}| + |\{ |\mathcal{H}_\epsilon g - \mathcal{H}_\epsilon f| > \alpha \}|
\]
By Lemma 5.3, the first term on the right hand side is at most \( \frac{C||f||_1}{t-\alpha} = \frac{C||g||_1}{t-\alpha} \). Thus we are done if we can show that the second term can be made arbitrarily small by making the intervals \( I_k \) smaller.

Note that
\[
\mathcal{H}_\epsilon f(x) = \frac{1}{\pi} \sum_{k=1}^{n} \left| g \cdot \chi_{I_k} \right|_1 \delta(x - a_k) \frac{\chi(x - a_k)}{x - a_k} = \frac{1}{\pi} \sum_{k=1}^{n} \int_{I_k} \frac{g(y) \chi(x - y)}{x - y} \ dy
\]
and
\[
\mathcal{H}_\epsilon g(x) = \frac{1}{\pi} \int_{\{|x-y| > \epsilon \} \cap \text{supp}(g)} \frac{g(y)}{x - y} \ dy = \frac{1}{\pi} \sum_{k=1}^{n} \int_{I_k} \frac{g(y) \chi(x - y)}{x - y} \ dy.
\]
Thus
\[
\pi |\mathcal{H}_\epsilon g(x) - \mathcal{H}_\epsilon f(x)| \leq \sum_{k=1}^{n} \int_{I_k} g(y) \left| \frac{\chi(x - y)}{x - y} - \frac{\chi(x - a_k)}{x - a_k} \right| \ dy
\]
Denote the right hand side by \( S(x) \). We will first show that \( S(x) \) is integrable. For this, assume that \( y \in I_k \) for some \( k \). Then we see that
\[
\int_{\mathbb{R}} \left| \frac{\chi(x - y)}{x - y} - \frac{\chi(x - a_k)}{x - a_k} \right| \ dy = \int_{A_{0,0}} 0 \ dy + \int_{A_{0,1}} \frac{1}{|x - a_k|} \ dy + \int_{A_{1,0}} \frac{1}{|x - y|} \ dy + \int_{A_{1,1}} \frac{|y - a_k|}{|x - y||x - a_k|} \ dy
\]
Where the set \( A_{i,j} \) with \( i, j \in \{0, 1\} \) corresponds to the set where \( \chi_t(x - y) \) and \( \chi_t(x - a_k) \) take the values \( i \) and \( j \) respectively. The first term, being zero, doesn’t concern us. The sets \( A_{0,1} \) and \( A_{1,0} \) are clearly intervals of length \( \min\{y - a_k, 2\epsilon\} \). If \( x \) is in the set \( A_{0,1} \) we must have that \( |x - a_k| > \epsilon \) and thus

\[
\int_{A_{0,1}} \frac{1}{|x - a_k|} \leq \frac{|A_{0,1}|}{\epsilon} \leq \frac{|y - a_k|}{\epsilon}.
\]

By the same reasoning

\[
\int_{A_{1,0}} \frac{1}{|x - y|} \leq \frac{|A_{1,0}|}{\epsilon} \leq \frac{|y - a_k|}{\epsilon}.
\]

For the last set, the function \( \frac{1}{|x - y||x - a_k|} \) doesn’t have any singularities in \( A_{1,1} \). Thus since it decays fast enough its integral over \( A_{1,1} \) must be bounded by some constant \( C_1 \), depending only on \( \epsilon \).

Collecting these estimates, we have that

\[
\int \frac{\chi_t(x - y)}{x - y} - \frac{\chi_t(x - a_k)}{x - a_k} \, dx \leq |y - a_k| \left( \frac{1}{\epsilon} + \frac{1}{\epsilon} + C_1 \right) \leq |I_k| \left( \frac{2}{\epsilon} + C_1 \right).
\]

For arbitrary \( \epsilon' > 0 \) we may choose the \( |I_k| \) so small that this is always less than \( \epsilon' \). Then an application of Fubini’s theorem shows that

\[
\int_R |S(x)| \, dx = \int_R \sum_{k=1}^n \int_{I_k} g(y) \left| \frac{\chi_t(x - y)}{x - y} - \frac{\chi_t(x - a_k)}{x - a_k} \right| \, dy \, dx
\]

\[
= \sum_{k=1}^n \int_{I_k} g(y) \int_R \left| \frac{\chi_t(x - y)}{x - y} - \frac{\chi_t(x - a_k)}{x - a_k} \right| \, dx \, dy
\]

\[
\leq \sum_{k=1}^n \int_{I_k} g(y) \epsilon' \, dy = \epsilon' \sum_{k=1}^n \|g \cdot \chi_{I_k}\| = \epsilon' \|g\|_1.
\]

Thus Markov’s inequality shows that \( \{|x \in \mathbb{R} : |S(x)| > \alpha\} \leq \frac{\|S\|_1}{\alpha} \leq \frac{\epsilon' \|g\|_1}{\alpha} \) which can be made arbitrarily small by a choice of \( \epsilon' \). Therefore we may make \( \{|\mathcal{H}_e g - \mathcal{H}_e f| > \alpha\} \) arbitrarily small and hence \( \{|\mathcal{H}_e g| > t\} \leq \frac{C \|g\|_1}{t - \alpha} \) for all \( \alpha > 0 \) less than \( t \). Letting \( \alpha \to 0 \) finishes the proof.

If now \( g(x) \in L^1 \) is real-valued and has compact support, we may write \( g(x) = g_+(x) - g_-(x) \) where \( g_+ \) and \( g_- \) are nonnegative and integrable as in Theorem 5.4 and \( \|g\|_1 = \|g_+\|_1 + \|g_-\|_1 \). Therefore

\[
\|\mathcal{H}_e g\|_{1,w} \leq 2(\|\mathcal{H}_e g_+\|_{1,w} + \|\mathcal{H}_e g_-\|_{1,w}) \leq 2C \|g\|_1,
\]

22
where $C$ is the constant in Theorem 5.4. In the same way we may divide a complex-valued compactly supported function $g$ into its real and imaginary parts, and apply the previous result to see that $||\mathcal{H}_c g||_{1,w} \leq 4C||g||_1$. This shows that the truncated Hilbert transforms satisfy the weak $(1,1)$ inequality for $L^1$ functions of compact support.

Assume now that $g \in C_0^\infty$. We know that $\mathcal{H} g(x)$ exists everywhere, but still need to prove that it satisfies the weak $(1,1)$ inequality. For this it is enough to show that

$$||\mathcal{H} g - \mathcal{H}_c g||_1 \to 0$$

as $\epsilon \to 0$, since if this is true we can first use Markov’s inequality to show that $||\mathcal{H} g - \mathcal{H}_c g||_{1,w} \to 0$ as $\epsilon \to 0$. Then as seen before, equation (5.1) would imply that $||\mathcal{H}_c g||_{1,w} \to ||\mathcal{H} g||_{1,w}$ as $\epsilon \to 0$, which would prove the weak $(1,1)$ inequality for $\mathcal{H} g$. It is thus enough to prove the following theorem.

**Theorem 5.5.** Let $1 \leq p < \infty$. If $g \in C_0^\infty$, then

$$||\mathcal{H} g - \mathcal{H}_c g||_p \to 0$$

as $\epsilon \to 0$.

**Proof.** Since $g$ is smooth with compact support, so is $g'$ and thus there exists a constant $K$ such that $|g'(x)| \leq K$ for all $x \in \mathbb{R}$. Then we may use the mean value theorem again to estimate that

$$|\mathcal{H} g(x) - \mathcal{H}_c g(x)| = \frac{1}{\pi} \lim_{\delta \to 0} \int_{\delta < |y| \leq \epsilon} \frac{g(x-y)}{y} \, dy$$

$$= \frac{1}{\pi} \int_{|y| \leq \epsilon} \frac{g(x-y) - g(x)}{y} \, dy$$

$$= \frac{1}{\pi} \int_{|y| \leq \epsilon} g'(\xi_y) \, dy \leq \frac{K}{\pi} \int_{|y| \leq \epsilon} \, dy = \frac{2\epsilon K}{\pi}$$

for all $x \in \mathbb{R}$. But for those $x$ that are further than $\epsilon$ away from $\text{supp}(g)$ we see that $\mathcal{H} g(x) - \mathcal{H}_c g(x) = 0$. Thus

$$\int_{\mathbb{R}} |\mathcal{H} g(x) - \mathcal{H}_c g(x)|^p \, dx \leq \frac{(2\epsilon K |S_\epsilon|)^p}{\pi^p},$$

where $S_\epsilon$ is the set of points with distance to $\text{supp}(g)$ less than $\epsilon$. But since $\text{supp}(g)$ is bounded, $S_\epsilon$ must be bounded too and thus the right hand side of (5.5) approaches zero as $\epsilon \to 0$. \qed

23
6  $L^2$ theory of the Hilbert transform

In this section we would like to prove that the Hilbert transform is strong $(2, 2)$. In fact much more holds, and our goal is to prove the following theorem.

Theorem 6.1. For $C^\infty_0$-functions the Hilbert transform is a $L^2$-isometry, meaning that $||\mathcal{H}f||_2 = ||f||_2$ for all $f \in C^\infty_0$.

A straight consequence of this theorem is that $\mathcal{H}$ satisfies the strong $(2, 2)$ inequality for $C^\infty_0$-functions, and an application of the continuous linear extension theorem would thus yield an unique extension to the whole $L^2$ that is also strong $(2, 2)$. In fact by continuity $||\mathcal{H}f||_2 = ||f||_2$ will also hold for a general $L^2$-function. We will now present a proof of this theorem using methods from complex analysis such as the results from Section 3, but we will also present an alternate proof using the Fourier transform afterwards.

Proof. First we fix a real-valued $f \in C^\infty_0$. Recall from Section 3 that $f + i\mathcal{H}f$ has an analytic extension $F(z)$ to the upper-half complex plane $\mathbb{H}$, given by

$$F(z) = F(x + iy) = \int_{\mathbb{R}} \frac{yf(t)}{(x-t)^2 + y^2} + \frac{i(x-t)f(t)}{(x-t)^2 + y^2} \, dt = \int_{\mathbb{R}} \frac{f(t)}{z-t} \, dt$$

for all $z \in \mathbb{H}$. We may extend $F$ to $\mathbb{R}$ by

$$F(x) = f(x) + i\mathcal{H}f(x)$$

for $x \in \mathbb{R}$. Then $F$ is a continuous function in $\mathbb{H} \cup \mathbb{R}$, which follows from the fact that $F$ is continuous in $\mathbb{H}$ and converges to $f + i\mathcal{H}f$ on the real line. Note that since we know that $F(z)$ is analytic in $\mathbb{H}$, so must be $F(z)^2$. We have for any $x \in \mathbb{R}$ that

$$F(x)^2 = (f(x) + i\mathcal{H}f(x))^2 = f(x)^2 - \mathcal{H}f(x)^2 + 2if(x)\mathcal{H}f(x). \quad (6.1)$$

The idea from now on is to prove that $F(x)^2$ has integral zero over the real line using an application of Cauchy’s theorem. This would show that its real part also has integral zero, which will turn out to imply that $||f||_2 = ||\mathcal{H}f||_2$.

Let us first use the fact that $f$ is in $C^\infty_0$ to choose $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$ and $a > 0$ such that $\text{supp}(f) \subset [-a, a]$. Fix also a $R > a$ arbitrarily large. For $y \geq 0$ we define $S_{y,R}$ to be the counterclockwise directed boundary of the rectangle with vertices $(-R, y), (R, y), (R, R + y)$ and $(-R, R + y)$ in the complex plane.
If \( y > 0 \) we see that \( S_{y,R} \) is contained in \( \mathbb{H} \), and since \( \mathbb{H} \) is simply connected the general version of Cauchy’s theorem implies that the integral of \( F(z)^2 \) over \( S_{y,R} \) is zero. We would now like to prove that the integral of \( F(z)^2 \) also vanishes over \( S_{0,R} \). This is relatively simple since the set

\[
\bigcup_{0 \leq y \leq 1} S_{y,R}
\]

is a compact subset of \( \mathbb{H} \), and thus \( F(z)^2 \) is uniformly continuous in it. We may calculate that

\[
\left| \int_{S_{0,R}} F(z)^2 \, dz \right| = \left| \int_{S_{0,R}} F(z)^2 \, dz - \int_{S_{y,R}} F(z)^2 \, dz \right|
\leq \int_{S_{0,R}} |F(z)^2 - F(z + iy)^2| \, dz,
\]

and this converges to zero as \( y \to 0 \) by uniform continuity.

Let us now divide \( S_{0,R} \) into two paths \( \gamma_R \) and \( \gamma_R' \) given by

\[
\gamma_R = [-R, R]
\]

and

\[
\gamma_R' = (R, R + iR] \cup [R + iR, -R + iR] \cup [-R + iR, -R).
\]

Observe that \( \gamma_R \) approaches the real line as \( R \to \infty \). Since the integral of \( F(z)^2 \) vanishes over \( S_{0,R} \) we have that

\[
\int_{\gamma_R} F(z)^2 \, dz = - \int_{\gamma_R'} F(z)^2 \, dz. \quad (6.2)
\]

Now we will prove that the integral of \( F(z)^2 \) over \( \gamma_R' \) approaches zero as \( R \to \infty \). This happens because if \( z \) lies on \( \gamma_R' \), then \( \text{dist}(z, \text{supp}(f)) \geq R - a \).
and thus
\[ |F(z)^2| = \left| \int_R \frac{f(t)}{z - t} dt \right|^2 \leq \left( \int_{\text{supp}(f)} \frac{|f(t)|}{|z - t|} dt \right)^2 \leq \frac{||f||^2}{(R-a)^2} \]
for all \( z \) on \( \gamma_R \), which implies that
\[ \left| \int_{\gamma_R} F(z)^2 dz \right| \leq \frac{||f||^2}{(R-a)^2} \int_{\gamma_R} dz = \frac{4R||f||^2}{(R-a)^2}, \]
Since this approaches zero as \( R \to \infty \), we have proven that
\[ \lim_{R \to \infty} \int_{\gamma_R} F(z)^2 dx = 0. \]
This implies that
\[ \int_R F(x)^2 dx = \lim_{R \to \infty} \int_{\gamma_R} F(z)^2 dz = - \lim_{R \to \infty} \int_{\gamma_R} F(z)^2 dz = 0, \quad (6.3) \]
which is what we wanted to prove. Now by (6.1) we have that \( \text{Re}(F(x)^2) = f(x)^2 - \mathcal{H} f(x)^2 \) for \( x \in \mathbb{R} \), (6.3) implies that
\[ 0 = \int_R \text{Re} F(z)^2 dx = \int_R (f(x)^2 - \mathcal{H} f(x)^2) dx = ||f||^2_2 - ||\mathcal{H} f||^2_2, \]
and thus \( ||f||^2_2 = ||\mathcal{H} f||^2_2 \) as wanted.

For the general case where \( f \) is complex-valued we may write \( f = f_1 + if_2 \), where \( f_1, f_2 \) are real-valued and in \( C^0_0 \). Then we apply the \( L^2 \)-isometry for \( f_1 \) and \( f_2 \) to see that
\[ ||\mathcal{H} f||^2_2 = \int_R |\mathcal{H} f(x)|^2 dx = \int_R (|\mathcal{H} f_1(x)|^2 + |\mathcal{H} f_2(x)|^2) dx \]
\[ = \int_R (|f_1(x)|^2 + |f_2(x)|^2) dx = ||f||^2_2, \]
which proves that \( \mathcal{H} \) is a \( L^2 \)-isometry for all \( f \in C^0_0 \).

6.1 The Fourier transform
There exists another relatively simple proof of Theorem 6.1 using the Fourier transform, and we will assume that the basic properties listed below are known to the reader. For proofs see [1].
Definition 6.1. The Fourier transform of a function \( f \in L^1 \) is denoted by \( \hat{f} \) and is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} \, dx.
\]

- The Fourier transform is linear and it is a \( L^2 \)-isometry on \( L^1 \cap L^2 \). Since \( L^1 \cap L^2 \) is dense in \( L^2 \), it may be extended to a linear isometry on \( L^2 \) by the continuous linear extension theorem.

- The convolution theorem states that
\[
\hat{f} \ast \hat{g} = \hat{f} \hat{g}
\]
for any \( f \) and \( g \) in \( L^1 \) or \( L^2 \).

To prove Theorem 6.1 again let us fix a \( f \in C_0^\infty \) and \( \epsilon > 0 \). We know that the truncated Hilbert transform \( \mathcal{H}_\epsilon \) may be written as a convolution
\[
\mathcal{H}_\epsilon f = h_\epsilon \ast f,
\]
where \( h_\epsilon(x) = \chi_\epsilon(x)/\pi x \). By the convolution theorem we have that
\[
\mathcal{H}_\epsilon f = \hat{h}_\epsilon \hat{f}.
\]

But what is \( \hat{h}_\epsilon \)? Note that since \( h_\epsilon \) is in \( L^2 \) the Fourier transform \( \hat{h}_\epsilon \) is well-defined. We notice that the functions \( h_\epsilon \chi_{[-R,R]} \) are in \( L^1 \cap L^2 \) for all \( R > 0 \), and since their limit in \( L^2 \) is \( h_\epsilon \), by the continuity of the Fourier transform we may calculate \( \hat{h}_\epsilon \) as the \( L^2 \)-limit
\[
\hat{h}_\epsilon = \lim_{R \to \infty} \int_{-R}^{R} h_\epsilon(x)e^{-2\pi i \xi x} \, dx.
\]

The expression inside the limit can be simplified, since
\[
\int_{-R}^{R} h_\epsilon(x)e^{-2\pi i \xi x} \, dx = \frac{1}{\pi} \int_{\epsilon}^{R} e^{-2\pi i \xi x} \frac{dx}{x} + \frac{1}{\pi} \int_{-R}^{-\epsilon} e^{-2\pi i \xi x} \frac{dx}{x} = \frac{1}{\pi} \int_{\epsilon}^{R} e^{-2\pi i \xi x} - e^{2\pi i \xi x} \frac{dx}{x} = \frac{-2i}{\pi} \int_{\epsilon}^{R} \frac{\sin(2\pi \xi x)}{x} \, dx.
\]

Now as \( R \to \infty \) these expressions converge pointwise almost everywhere to
\[
-2i \int_{\epsilon}^{\infty} \frac{\sin(2\pi \xi x)}{x} \, dx, \tag{6.4}
\]
which is known to exist for all $\xi$. This implies that (6.4) is also the $L^2$-limit of the Fourier transforms of $h_\epsilon \chi_{[-R,R]}$ since every converging sequence of functions in $L^2$ has a subsequence converging pointwise almost everywhere. Thus $\hat{h}_\epsilon(\xi)$ is the expression (6.4). It is easy to see that each $\hat{h}_\epsilon$ is bounded by some constant independent of $\epsilon$, and that pointwise we have that

$$\lim_{\epsilon \to 0} \hat{h}_\epsilon(\xi) = \frac{-2i}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin(2\pi \xi x)}{x} \, dx$$

$$= \frac{-2i}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin(2\pi \mathrm{sgn}(\xi)|\xi| x)}{x} \, dx$$

$$= \frac{-2i\mathrm{sgn}(\xi)}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin(2\pi |\xi| x)}{x} \, dx$$

$$= \frac{-2i\mathrm{sgn}(\xi)}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon'}^{\infty} \frac{\sin(x)}{x} \, dx$$

(Where $\epsilon' = 2\pi |\xi| \epsilon$)

$$= \frac{-2i\mathrm{sgn}(\xi) \pi}{2} = -i\mathrm{sgn}(\xi).$$

Now we may apply Theorem 5.5 to deduce that $\mathcal{H}_\epsilon f \to \mathcal{H} f$ in $L^2$, and by the continuity of the Fourier transform on $L^2$ we see that if $f \in C_0^\infty$,

$$\mathcal{H} f = (\lim_{\epsilon \to 0} \mathcal{H}_\epsilon f) = \lim_{\epsilon \to 0} (\mathcal{H}_\epsilon f) = \lim_{\epsilon \to 0} (\hat{h}_\epsilon \hat{f}),$$

where the limits are taken in $L^2$. Since $\hat{f}$ is in $L^2$ and the expressions $\hat{h}_\epsilon$ are uniformly bounded and converge pointwise to $-i\mathrm{sgn}(\xi)$, we may apply the dominated convergence theorem to see that

$$\lim_{\epsilon \to 0} (\hat{h}_\epsilon(\xi) \hat{f}(\xi)) = -i\mathrm{sgn}(\xi) \hat{f}(\xi),$$

where the limit is again in $L^2$. This shows that

$$\mathcal{H} f(\xi) = -i\mathrm{sgn}(\xi) \hat{f}(\xi).$$

Especially since the Fourier transform is a $L^2$-isometry we have that

$$||\mathcal{H} f||_2 = ||\mathcal{H} f||_2 = ||-i\mathrm{sgn}(\xi) \hat{f}||_2 = ||\hat{f}||_2 = ||f||_2$$

as desired. \[\square\]
The Marcinkiewicz interpolation theorem

In this section we will complete our goal of extending the Hilbert transform to a bounded operator on $L^p$ for $1 < p < \infty$. Let us first state it as a theorem to be proven.

**Theorem 7.1.** The Hilbert transform $\mathcal{H}$ defined in $C_0^\infty$ via equation (2.1) may be extended uniquely to a bounded linear operator on $L^p$ for $1 < p < \infty$.

In the previous sections we have only proven the special case $p = 2$ and found an additional weaker bound on $L^1$. For a reader not familiar with the following theorem by Marcinkiewicz it might thus seem like we are still far away from our goal of proving Theorem 7.1. But the Marcinkiewicz interpolation theorem tells us that we can combine the previous two estimates to prove the strong bound on $L^p$ for $1 < p < 2$, leaving us only with the case $2 < p < \infty$ that we finish with a duality argument.

We will first however turn to the question of how to define the Hilbert transform of a $L^p$ function for $1 < p < 2$. Having defined it for $L^1$ and $L^2$ functions, we note that the Hilbert transform may also be defined on $L^1 + L^2$, since if $f = f_1 + f_2$ with $f_1 \in L^1$ and $f_2 \in L^2$ we may define its Hilbert transform as $\mathcal{H} f = \mathcal{H} f_1 + \mathcal{H} f_2$. We must check that this definition is independent of the choice of $f_1$ and $f_2$, since we have given two different definitions for the Hilbert transform on $L^1$ and $L^2$, both using the continuous linear extension theorem. We first note that it is enough to check that these definitions coincide on $L^1 \cap L^2$.

Let now $f \in L^1 \cap L^2$ be given, and denote by $\mathcal{H}_1 f$ and $\mathcal{H}_2 f$ the Hilbert transforms of $f$ given by the definitions in $L^1$ and $L^2$ respectively. We may now choose a sequence $(f_n) \subset C_0^\infty$ of functions converging to $f$ in both $L^1$ and $L^2$. By definition $\mathcal{H} f_n$ converges to $\mathcal{H}_1 f$ in $L^1$ and to $\mathcal{H}_2 f$ in $L^2$. Fixing now $\epsilon > 0$ we see that

$$\left|\{||\mathcal{H}_1 f - \mathcal{H}_2 f|| > \epsilon\}\right| \leq \left|\{||\mathcal{H}_1 f - \mathcal{H} f_n|| > \epsilon/2\}\right| + \left|\{||\mathcal{H}_1 f_n - \mathcal{H}_2 f|| > \epsilon/2\}\right|$$

$$\leq \frac{2||\mathcal{H}_1 f - \mathcal{H} f_n||_{1,\infty}}{\epsilon} + \frac{2||\mathcal{H} f_n - \mathcal{H}_2 f||_2}{\epsilon}.$$ 

And the utmost right hand side approaches zero as $n \to \infty$. We conclude that the set $\{||\mathcal{H}_1 f - \mathcal{H}_2 f|| > \epsilon\}$ has measure zero for all $\epsilon > 0$, and this implies that $\mathcal{H}_1 f = \mathcal{H}_2 f$ almost everywhere as wanted.

Let us assume now that $f$ is an arbitrary $L^p$ function where $1 < p < 2$.
For any $\alpha > 0$, we can decompose $f$ as

$$f(x) = f(x)\chi_{\{|f(x)|>\alpha\}}(x) + f(x)\chi_{\{|f(x)|\leq \alpha\}}(x) = f_1^\alpha(x) + f_2^\alpha(x).$$

We see that $f_1^\alpha$ is in $L^1$ since $1 - p < 0$ and thus

$$\int_{\mathbb{R}} |f_1^\alpha| dx = \int_{|f(x)|>\alpha} |f|^p |f|^{1-p} dx \leq \alpha^{1-p} \int_{|f(x)|>\alpha} |f|^p dx \leq \alpha^{1-p} ||f||_p.$$ 

In the same way we see that $f_2^\alpha$ is in $L^2$. Hence in this way the Hilbert transform is defined for any $f \in L^p$, and we can return to proving its boundedness. For this we will now state the Marcinkiewicz interpolation theorem in its full glory.

**Theorem 7.2.** (The Marcinkiewicz interpolation theorem). Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces. Let $1 \leq p_0 < p_1 \leq \infty$ and assume that we are given a sublinear operator $T$ from $L^{p_0}(X) + L^{p_1}(X)$ to the measurable functions on $Y$ such that $T$ is both weak $(p_0, p_0)$ and weak $(p_1, p_1)$. Then $T$ is strong $(p, p)$ for $p_0 < p < p_1$.

We will not prove the whole theorem here, but will instead only consider the special case relevant to us where $p_0 = 1$, $p_1 = 2$, $X = Y = \mathbb{R}$ and $T = \mathcal{H}$ is the Hilbert transform. It should be noted that the proof we will give is not hard to generalize to prove the whole theorem, but being a special case it should be easier to follow. A full proof can be found in [1]. For the proof we first need the following concept.

**Definition 7.1.** The distribution function $D_f : [0, \infty) \to [0, \infty]$ of a measurable function $f$ is defined as

$$D_f(\alpha) = |\{x \in \mathbb{R} : |f(x)| > \alpha\}|.$$

**Lemma 7.3.** We may calculate the $L^p$-norm of a function $f$ using its distribution function by the equality

$$||f||_p^p = \int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty \alpha^{p-1} D_f(\alpha) d\alpha.$$ 

**Proof.** A straightforward calculation using Fubini’s theorem shows that

$$\int_0^\infty p \alpha^{p-1} D_f(\alpha) d\alpha = \int_0^\infty p \alpha^{p-1} \left( \int_{|f(x)|>\alpha} dx \right) d\alpha$$

$$= \int_0^\infty p \alpha^{p-1} \left( \int_{|f(x)|>\alpha} \chi_{\{|f(x)|>\alpha\}}(x) dx \right) d\alpha$$

$$= \int_{\mathbb{R}} \int_0^\infty p \alpha^{p-1} \chi_{\{|f(x)|>\alpha\}}(x) d\alpha dx$$

$$= \int_{\mathbb{R}} \int_0^{|f(x)|} p \alpha^{p-1} d\alpha dx = \int_{\mathbb{R}} |f(x)|^p dx.$$
We now begin the proof of our special case of the interpolation theorem by fixing a $f \in L^p$ for $1 < p < 2$ and decomposing it as $f = f_1^\alpha + f_2^\alpha$ for each $\alpha > 0$. Thus we may use the weak (1,1) and (2,2) inequalities to get the estimate

$$D_{\mathcal{H}} f(\alpha) \leq D_{\mathcal{H}} f_1^\alpha(\alpha/2) + D_{\mathcal{H}} f_2^\alpha(\alpha/2) \leq \frac{C_1 ||f_1^\alpha||_1}{\alpha} + \frac{C_2 ||f_2^\alpha||_2^2}{\alpha^2}$$

for some constants $C_1$ and $C_2$. Using Lemma 7.3 with this estimate yields that

$$||\mathcal{H} f||_p^p = p \int_0^\infty \alpha^{p-1} D_{\mathcal{H}} f(\alpha) d\alpha$$

$$\leq pC_1 \int_0^\infty \alpha^{p-2} ||f_1^\alpha||_1 d\alpha + pC_2 \int_0^\infty \alpha^{p-3} ||f_2^\alpha||_2^2 d\alpha.$$

We now simplify the two terms on the utmost right hand side using Fubini’s theorem. For the first one,

$$pC_1 \int_0^\infty \alpha^{p-2} ||f_1^\alpha||_1 d\alpha = pC_1 \int_0^\infty \alpha^{p-2} \int_\mathbb{R} |f(x)| \chi_{\{|f(x)| > \alpha\}} dx d\alpha$$

$$= pC_1 \int_\mathbb{R} \int_0^\infty \alpha^{p-2} |f(x)| \chi_{\{|f(x)| > \alpha\}} dx d\alpha$$

$$= pC_1 \int_\mathbb{R} |f(x)| \int_0^{\lfloor f(x) \rfloor} \alpha^{p-2} d\alpha dx$$

$$= pC_1 \int_\mathbb{R} |f(x)| \frac{|f(x)|^{p-1}}{p-1} dx$$

$$= \frac{pC_1}{p-1} ||f||_p^p.$$

And similarly for the second one,

$$pC_2 \int_0^\infty \alpha^{p-3} ||f_2^\alpha||_2^2 d\alpha = pC_2 \int_0^\infty \alpha^{p-3} \int_\mathbb{R} |f(x)|^2 \chi_{\{|f(x)| \leq \alpha\}} dx d\alpha$$

$$= pC_2 \int_\mathbb{R} \int_0^\infty \alpha^{p-3} |f(x)|^2 \chi_{\{|f(x)| \leq \alpha\}} dx d\alpha$$

$$= pC_2 \int_\mathbb{R} |f(x)|^2 \int_0^{\lfloor f(x) \rfloor} \alpha^{p-3} d\alpha dx$$

$$= pC_2 \int_\mathbb{R} |f(x)|^2 \frac{|f(x)|^{p-2}}{2-p} dx$$

$$= \frac{pC_2}{2-p} ||f||_p^p.$$
Combining these estimates we get that
\[
||\mathcal{H}f||_p^p \leq \left( \frac{pC_1}{p-1} + \frac{pC_2}{2-p} \right) ||f||_p^p,
\]
proving the theorem. \qed

7.1 Boundedness of the Hilbert transform using duality

In this subsection we will consider \( p \) as fixed with \( 2 < p < \infty \), and prove the remaining case in Theorem 7.1. Let us denote by \( p' \) the number such that \( 1/p + 1/p' = 1 \), especially \( 1 < p' < 2 \). We would first like to prove the following lemma

**Lemma 7.4.** If \( f \in L^p \), then there exists a sequence \( (g_n) \) of \( C_0^\infty \)-functions such that
\[
||g_n||_{p'} = 1
\]
for all \( n \) and
\[
||f||_p = \lim_{n \to \infty} \left| \int_R f(x)g_n(x)dx \right|. \tag{7.2}
\]

*Proof.* Without loss of generality assume that \( ||f||_p > 0 \). Then we define
\[
g(x) = \text{sgn}(f(x)) \left( \frac{|f(x)|}{||f||_p} \right)^{p-1}.
\]
We see that \( g \) has the following properties:
\[
||g||_{p'} = 1 \tag{7.3}
\]
and
\[
||f||_p = \int_R f(x)g(x)dx. \tag{7.4}
\]
We now use the denseness of \( C_0^\infty \) in \( L^{p'} \) to choose a sequence \( (g_n) \subset C_0^\infty \) such that \( ||g - g_n||_{p'} \to 0 \) as \( n \to \infty \). Since this implies that \( ||g_n||_{p'} \to ||g||_{p'} = 1 \) as \( n \to \infty \) we see that
\[
||g - \frac{g_n}{||g_n||_{p'}}||_{p'} \to 0
\]
as \( n \to \infty \). Therefore to satisfy (7.1) we will choose \( g_n^*(x) = g_n/||g_n||_{p'} \) as the sequence sought in the lemma. It remains to prove that (7.2) holds. For this we use (7.4) and Hölder’s inequality to calculate that
\[
\left| ||f||_p - \int_R f(x)g_n^*(x)dx \right| = \left| \int_R f(x)(g(x) - g_n^*(x))dx \right| \leq ||f||_p ||g - g_n^*||_{p'}.
\]
This approaches zero as \( n \to \infty \), finishing the proof. \qed
Another useful fact is that for $C_0^\infty$-functions we have the following identity

$$\int_\mathbb{R} \mathcal{H} f(x)g(x)dx = -\int_\mathbb{R} f(x)\mathcal{H} g(x)dx.$$  \hfill (7.5)

This can be established by first calculating for each $\epsilon > 0$ that

$$\int_{-\infty}^{\infty} \mathcal{H}_\epsilon f(x)g(x)dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x-t)\chi_\epsilon(t)g(x)}{t}dt dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)\chi_\epsilon(t)g(x+t)}{t} dt dx = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)\chi_\epsilon(t)g(x-t)}{t}dt dx = -\int_{-\infty}^{\infty} f(x)\mathcal{H}_\epsilon g(x)dx,$$

and applying dominated convergence to conclude that equality holds even if we let $\epsilon \to 0$.

Let us now fix a $f \in C_0^\infty$. As we may write

$$\mathcal{H} f(x) = \frac{1}{\pi} \int_{|t|\geq 1} \frac{f(x-t)}{t} dt + \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|>\epsilon} \frac{f(x-t) - f(x)}{t} dt,$$

we see that $\mathcal{H} f$ is bounded by $M = C(||xf||_\infty + ||f'||_\infty)$ for a large enough constant $C$. Then since it belongs to $L^2$ it must also belong to $L^p$ since $p > 2$ and thus

$$\int_\mathbb{R} |f(x)|^p dx = \int_\mathbb{R} |f(x)|^2 |f(x)|^{p-2} dx \leq M^{p-2} \int_\mathbb{R} |f(x)|^2 dx \leq M^{p-2}||f||^2_2.$$

Therefore we may use Lemma 7.4 and (7.5) to calculate that

$$||\mathcal{H} f||_p = \lim_{n \to \infty} \int_\mathbb{R} \mathcal{H} f(x)g_n(x)dx \quad \text{(For a sequence } (g_n) \subset C_0^\infty)$$

$$= \lim_{n \to \infty} \left| \int_\mathbb{R} f(x)\mathcal{H} g_n(x)dx \right|$$

$$\leq \lim_{n \to \infty} ||f||_p ||\mathcal{H} g_n||_{p'} \quad \text{(By Hölder’s inequality)}$$

$$\leq C||f||_p,$$

where the constant $C$ is as in the strong $(p', p')$ inequality. Again by the continuous linear extension theorem we may extend the Hilbert transform to a continuous linear transformation from $L^p$ to itself. This concludes the proof of Theorem 7.1. \hfill \Box
References
