HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

University of Helsinki
Department of Mathematics and Statistics

Topology I
Final examination, 13. 5. 2008

1. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, put

$$x * y = (|x_1| + |x_2|)(|y_1| + |y_2|),$$
$$\|x\| = \sqrt{x * x}$$

and

$$d(x,y) = \begin{cases} \|x\| + \|y\|, & \text{for } x \neq y \\ 0, & \text{otherwise.} \end{cases}$$

a) Is it true that $*$ is an inner product of the vector space $\mathbb{R}^2$?
b) Is $\|\cdot\|$ a norm of the vector space $\mathbb{R}^2$?
c) Is d a metric of the set $\mathbb{R}^2$?

2. Find an example of a subset A of the plane $\mathbb{R}^2$ such that $\emptyset \neq \operatorname{int} A \neq A \neq \bar{A}$.

3. Determine which of the sets

$$A = \{\, (x, 0) \mid x \in [0, 2\pi] \,\},$$
$$B = S^1 = \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\} \text{ ja}$$
$$C = \{\, (x, y) \in S^1 \mid y \geq 0 \,\}$$

are homeomorphic with each other.

4. Define the concepts of a Cauchy sequence and completeness in metric spaces. Show that a convergent sequence is a Cauchy sequence.

5. Consider $A = \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 = 1 \,\}$ and a mapping $f\colon A \to \mathbb{R}$, $f(x) = xe^y$.
a) Show that A is compact.
b) Show that f attains its largest and smallest value on the set A.

[The continuity of the exponential function may be assumed known.]

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

University of Helsinki
Department of Mathematics and Statistics
Topology I
Final examination, 12. 6. 2008

1. Consider the plane set

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1 \}.$$

Find the diameter of A relative to the metric a) induced by the sup-norm $\|(x,y)\|_0 = \max\{|x|, |y|\}$, b) induced by the ℓ_1-norm $\|(x,y)\|_1 = |x| + |y|$.

2. Let (X, d) be a metric space and let $A \subset X$. Prove that $\partial\partial A \subset \partial A$.

3. Put $f \colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^5 + 2}{10}.$$

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_0 = 0$ and $a_{n+1} = f(a_n)$, for all $n \in \mathbb{N}$. Prove with the aid of Banach fixed point theorem that the limit $a = \lim_{n\to\infty} a_n$ exists and satisfies the equation $a^5 - 10a + 2 = 0$.

4. Find an example of two continuous mappings $f \colon A \to \mathbb{R}$ and $g \colon B \to \mathbb{R}$ such that
a) $A \subset \mathbb{R}^2$ is closed, but the image $f[A]$ is not closed in $\mathbb{R}$,
b) $B \subset \mathbb{R}^2$ is bounded, but the image $g[B]$ is unbounded.
In cases a and b, is it possible to use the same example, i.e., can you choose $f = g$ and $A = B$?

5. Denote $f \colon [-2, 2] \times [-2, 2] \to \mathbb{R}$,

$$f(x, y) = xy(x^3 + y^2).$$

Provided it is known that the domain $[-2, 2] \times [-2, 2]$ of f is connected, prove that the equation $f(x, y) = \pi$ has a solution.

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

University of Helsinki
Department of Mathematics and Statistics
Topology I
Final examination, 14 Aug, 2008

1. Let E be an inner product space and $x, y \in E$. Prove the parallelogram rule

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

2. Does there exist a set $A \subset \mathbb{R}$ with $\partial\partial A \neq \partial A$?

3. Find an example of two subsets of the plane which are not homeomorphic.

4. a) Formulate the Banach fixed point theorem and explain the concepts used in the formulation.

b) Put $f\colon [1,2] \to [1,2]$,

$$f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right),$$

and consider the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_0 = 2$ and $x_{n+1} = f(x_n)$, for every $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Show that $a = \lim_{n\to\infty} x_n$ exists and determine a.

5. Which of the sets

$$A = \{\,(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 = 1\,\},$$
$$B = \{\,(x,y) \in \mathbb{R}^2 \mid x^2 - 2y^2 = 1\,\} \text{ and}$$
$$C = \{\,(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 < 1\,\}$$

are compact?