Matematiikan ja tilastotieteen laitos

Topologiset transformaatioryhmät II

Loppukoe

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1. Prove the following result. Let G be a compact topological group, and let $g_0 \in G$. We denote

$$A = \{ g_0^n \mid n \geq 0 \}.$$

Then $\bar{A}$ is a closed subgroup of G.

2. Let G be a compact topological group, which is a first countable space, i.e., an N_1-space. Let $X = (X, d)$ be a metric space, and assume that G acts on the space X by a continuous action. We consider it to be known that $\hat{d} : X \times X \to \mathbb{R}$, defined by

$$\hat{d}(x,y) = \int_G d(gx, gy)\, dg \quad , \text{ for every } (x,y) \in X \times X,$$

is a G-invariant metric on the set X. Prove that the spaces (X, d) and $(X, \hat{d})$ are homeomorphic.

3. Formulate and prove the Tietze-Gleason extension theorem.

4. Prove the following result. Let G be a topological group which acts continuously on a topological space X. If A is a compact subset of X and B is a closed subset of X, then

$$G(A|B) = \{ g \in G \mid A \cap gB \neq \phi \}$$

is a closed subset of G.

5. Let X be a G-space, where G is a locally compact group and X is a locally compact Hausdorff space. Suppose that for any $x, y \in X$ there exist neighborhoods B_x and B_y, of x and y, respectively, in X, such that $\overline{G(B_y, B_x)}$ is compact. Prove that the action of G on X is proper. (Here "proper" means "Borel proper".)