Matematiikan laitos
Topological transformation groups II
Loppukoe
20.5.2003

1. Prove the following result. Suppose that G is a compact topological group, and let $g_0 \in G$. Let
$$A = \{g_0^n | n \geq 0\}.$$
Then $\overline{A}$ is a subgroup of G.

2. Formulate and prove the Tieze–Gleason extension theorem.

3. Prove the following result. Let X be a G–space, where G is a compact topological group, and assume that the topological space X is metrizable. Then there exists a G–invariant metric $\hat{d}$ for X.

4. Prove the following result. Suppose that a topological group G acts continuously on a topological space X. If A is a compact subset of X and B is a closed subset of X, then
$$G(B|A) = \{g \in G | B \cap gA \neq \emptyset\}$$
is a closed subset of G.

Matematiikan ja tilastotieteen laitos
Topologiset transformaatioryhmät II
Loppukoe
10.8.2004

1. Prove the following result. Let X be a G–space, and let $H \subset G$ be a closed subgroup of G. Then

$$\Psi: W(H) \times X^H \to X^H$$

$$(nH, x) \longmapsto nx$$

is a well–defined and continuous action of $W(H) = N(H)/H$ on X^H.

2. Prove the following result. Let X be a Hausdorff G–space, where G is a compact topological group, and let $x \in X$. Then the orbit Gx is a G–invariant closed subset of X and there exists a G–equivariant homeomorphism

$$f: Gx \to G/G_x.$$

3. Formulate and prove the Tietze–Gleason extension theorem.

4. Prove the following result. Let G be a locally compact topologial group and let X be a Hausdorff space, and let $\Phi: G \times X \to X$ be a continuous action of G on X. Then the following conditions are equivalent.
(i) $\Phi: G \times X \to X$ is a proper action of G on X.
(ii) The map $\Phi^*: G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$, is a proper map.

Matematiikan ja tilatotieteen laitos
Topologiset transformaatioryhmät II
Loppukoe
20.1.2005

1. Prove the following result. Suppose G is a compact topological group, and let $g_0 \in G$. We denote
$$A = \{g_0^n \mid n \geq 0\}.$$
Then $\overline{A}$ is a closed subgroup of G.

2. Let G be a compact topological group which is a first countable space, i.e., an N_1–space. Let $X = (X, d)$ be a metric space, and assume that G acts on X by a continuous action. We consider it known that $\hat{d}\colon X \times X \to \mathbb{R}$, defined by
$$\hat{d}(x, y) = \int_G d(gx, gy)dg, \text{ for every } \ (x, y) \in X \times X,$$
is a G–invariant metric on the set X. Prove that the spaces (X, d) and $(X, \hat{d})$ are homeomorphic,

3. Prove the following result. Let G be a topological group which acts continuously on a topological space X. If A is a compact subset of X and B is a closed subset of X, then
$$G(A|B) = \{g \in G \mid A \cap gB \neq \emptyset\}$$
is a closed subset of G.

4. Let G be a locally compact topological group which acts on a locally compact Hausdorff space X by a proper action $\Phi\colon G \times X \to X$. Prove that:
a) The map $\Phi|\colon G \times A \to X$ is a proper map, for each compact subset A of X.
b) The map $\varphi_x\colon G/G_x \to Gx,\ gG_x \mapsto gx$, is a homeomorphism, for every $x \in X$.

Matematiikan ja tilastotieteen laitos
Topologiset transformaatioryhmät II
Loppukoe
24.5.2005

1. Let G be a compact topological group, and let H be a closed subgroup of G. Suppose that $g \in G$ is such that $gHg^{-1} \subset H$. Show that $gHg^{-1} = H$, i.e., show that $g \in N(H)$.

2. a) Let G be a topological group, and let X be a G–space, where X is Hausdorff. Let H be a closed subgroup of G. Show that the fixed point set of H, X^H, is a closed subset of X.
b) Let $W(H) = N(H)/H$. Show that there is, a well–defined and continuous induced action of $W(H)$ on X^H.

3. Let G be a topological group and let H and K be closed subgroups of G. Show that G/H and G/K are G–homeomorphic if and only if H and K are conjugate.

4. Let G be a compact topological group and let X be a G–space. Suppose that X is metrizable and let $d\colon X \times X \to \mathbb{R}$ be a metric for X. Define $\hat{d}\colon X \times X \to \mathbb{R}$ by

$$\hat{d}(x, y) = \sup\{d(gx, gy) | g \in G\},$$

for all $(x, y) \in X \times X$. Show that $\hat{d}$ is a G–invariant metric for the topological space X.

5. Let X be a G–space, where G is a locally compact group and X is a locally compact Hausdorff space. Suppose that for any $x, y \in X$ there exist neighborhoods B_x and B_y, of x and y respectively, in X such that $\overline{G(B_y|B_x)}$ is compact. Show that the action of G on X is proper. (Here "proper" means "Borel proper".)

Matematiikan ja tilastotieteen laitos
Topologiset transformaatioryhmät II
Loppukoe
15.6.2005

1. Prove the following result. Suppose G is a compact topological group, and let $g_0 \in G$. We denote
$$A = \{g_0^n \mid n \geq 0\}.$$
Then $\overline{A}$ is a closed subgroup of G.

2. Let G be a compact topological group which is a first countable space, i.e., an N_1–space. Let $X = (X, d)$ be a metric space, and assume that G acts on X by a continuous action. We consider it known that $\hat{d}: X \times X \to \mathbb{R}$, defined by
$$\hat{d}(x, y) = \int_G d(gx, gy)dg, \text{ for every } (x, y) \in X \times X,$$
is a G–invariant metric on the set X. Prove that the spaces (X, d) and $(X, \hat{d})$ are homeomorphic,

3. Prove the following result. Let G be a topological group which acts continuously on a topological space X. If A is a compact subset of X and B is a closed subset of X, then
$$G(A|B) = \{g \in G \mid A \cap gB \neq \emptyset\}$$
is a closed subset of G.

4. Let G be a locally compact topological group and let X be a Hausdorff space, and let $\Phi: G \times X \to X$ be a continuous action of G on X. Prove that the following conditons are equivalent.
(i) $\Phi: X \to X$ is a proper action of G on X.
(ii) The map $\Phi^*: G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is a proper map.
(iii) The map $\Phi|: G \times A \to X$ is a proper map, for each compact subset A of X.

5. Prove the following result. Let G be a locally compact topological group, and let X be a locally compact proper G–space. Then the orbit space X/G is Hausdorff and locally compact.