Matematiikan ja tilastotieteen laitos
Topologiset transformaatioryhmät I
Loppukoe
3.4.2006

1. Prove the following result. Let G be a topological group and let G_0 be the identity component of G. Then the topological group G/G_0 is totally disconnected.

2. Give a complete proof of the following result. Let G be a connected, locally compact, topological group. Then G is a countable union of compact sets.

3. Prove the following result. Let G be a compact topological group, and let $f : G \to \mathbb{R}$ be a continuous function such that $f(g) \geq 0$, for all $g \in G$, and $f \neq 0$. Then

$$\int f(g)dg > 0.$$

4. Prove the following result. Let G be a compact topological group, and let $f : G \to \mathbb{R}^n$ be a continuous map. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. The the Haar integral of $T \circ f : G \to \mathbb{R}^m$ is given by

$$\int (T \circ f)dg = T(\int f(g)dg).$$

5. Give the main parts of the proof of the following result. Let G be a compact topological group, and let $\varphi : G \to \mathrm{GL}(n;\mathbb{R})$ be a linear representation of G. [Here $\mathrm{GL}(n;\mathbb{R})$ denotes the topological group of all non–singular $(n \times n)$–matrices over $\mathbb{R}$, $n \geq 1$]. Then φ is equivalent to an orthogonal representation of G.

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1. Prove the following result. Let G be a topological group, and let H be a closed normal subgroup of G. Then G/H is a topological group.

2. Prove the following result. Let G be a locally connected topological group. Then G/G_0 is discrete.

3. Prove the following result. Let G and H be compact topological groups, and let $f\colon G \times H \to \mathbb{R}$ be a continuous function. Then

$$\int_H (\int_G f(g,h)dg)dh = \int_G (\int_H f(g,h)dh)dg.$$

4. Prove the following result. Let V be a linear representation space for a compact topological group G and let W be a G–invariant linear subspace of V. Then there exists a G–invariant linear space U of V such that $V = U \oplus W$.

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1. Prove the following result. Let G be a topological group and let H be a closed subgroup of G. Then the quotient space G/H is Hausdorff.

2. Give the definition of the orthogonal group $\mathrm{O}(n)$, and prove that $\mathrm{O}(n)$ is a compact subgroup of $\mathrm{GL}(n;\mathbb{R})$.

3. Prove the following result. Let G be a compact topological group. Suppose $f\colon G \to \mathbb{R}^n$ is a continuous map, and let $T\colon \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Then the Haar integral of $T \circ f\colon G \to \mathbb{R}^k$ is given by

$$\int (T \circ f)(g)dg = T(\int f(g)dg).$$

4. Let G be a compact topological group and let $\varphi\colon G \to \mathrm{GL}(n;\mathbb{R})$ be a linear representation of G. Define $\Phi\colon G \to \mathrm{GL}(\mathbb{R}^n)$ by the condition that $[\Phi(g)]_{e,e} = \varphi(g)$, for each $g \in G$. Show that there exists an inner product $\langle,\rangle$ in $\mathbb{R}^n$ such that

$$\Phi(g)\colon (\mathbb{R}^n, \langle,\rangle) \to (\mathbb{R}^n, \langle,\rangle)$$

is an orthogonal map, for every $g \in G$.