# STOCHASTIC MODELLING
# FINAL EXAM (24.11.2008)

**Problem 1.** Suppose that the life-time $T$ of an electrical component is a discrete random variable

$$P(T = k) = a_k, \text{ for } k = 1, 2, 3, \ldots.$$

At each time step $t = 1, 2, 3\ldots$ the electrical component is checked and if it is found broken then it is replaced immediately with a new one. Let $X_t \in \{0, 1, 2, \ldots\}$ be the elapsed life-time of the (current) component at time step $t$, with the convention that if the previous component broke at time $t$ then $X_t = 0$ to denote that the new one has just been substituted for the broken one. (Also $X_0 = 0$.) a) Find the transition probabilities of the Markov chain $(X_t)_{t \geq 0}$, and b) find the expected proportion of time that the chain $(X_t)_{t \geq 0}$ spends in state 0.

**Problem 2.** Consider the lattice below

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  B
A X B
A Y Z B
A A A
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Suppose that a discrete time Markov chain is started from one of the inner points of the lattice ($X, Y$ or $Z$). This Markov chain moves at each time step one step either up, down, left or right, all with equal probability $\frac{1}{4}$. For each of the three possible starting points ($X, Y$ or $Z$) calculate the probability that the chain will reach one of the border points marked with A before it reaches any of the border points marked with B.

**Problem 3.** Show that a transition matrix $P = (p_{ij})$ of a finite state space Markov chain has a uniform distribution as its stationary distribution if and only if all columns of $P$ sum up to 1, i.e. if $\sum_{i=1}^{n} p_{ij} = 1$ for all $j = 1, \ldots, n$, where $n$ is the size of the state space.

**Problem 4.** Suppose that you have access to a sample $(u_1, \ldots, u_n)$ of computer-generated random values from the uniform distribution on the interval $[0, 1]$. How would you use that sample to approximate the integral $\int_0^1 x^5 dx$? (Give also a mathematical argument to validate the method.)

**Problem 5.** Ms. A went fishing yesterday. Suppose that she catches fish according to a Poisson process with an average of 2 fishes per hour. This time she got 3 fishes in 3 hours. What is the probability that all those 3 fishes were caught within the last hour?

(Remember: If $X \sim \text{Poisson}(\lambda)$ then $P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, for $k = 0, 1, 2, \ldots$)