Statistical inference - Exam August 9, 2007 (Casella&Berger)

1. (a) Explain what is meant by a sufficient statistic (definition, meaning, interpretation).

(b) Suppose that $X_1, ..., X_n$ is a random sample from an exponential distribution with probability density function $f(x) = (1/\theta)\exp(-x/\theta)$ $(x > 0, \theta > 0)$. Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for the parameter θ.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a Bernoulli(p) distribution, that is, $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$ where $0 < p < 1$ is assumed.

(a) Derive the likelihood function of the parameter p.

(b) Derive the maximum likelihood estimator of the parameter p.

(c) Show that the maximum likelihood estimator attains the Cramér-Rao Lower Bound.

3. Let $X_1, ..., X_n$ be a random sample from a distribution whose probability density function is

$$f(x;\theta) = 2\theta x \exp\left(-\theta x^2\right), \quad x > 0, \quad \theta > 0.$$

Derive the likelihood ratio test for testing H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$.

4. Let $X_1, ..., X_n$ be a random sample from a $n(\mu, 1)$ distribution (normal distribution with expectation μ and variance 1). Derive a confidence interval for the expectation μ.

Note: If the random variable X has a $n(\mu, \sigma^2)$ distribution its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

Statistical inference - Exam May 13th, 2008 (Casella&Berger)

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1. (a) Explain what is meant by a sufficient statistic (definition, meaning, interpretation).

(b) Suppose that $X_1, ..., X_n$ is a random sample from an exponential distribution with probability density function $f(x) = (1/\theta)\exp(-x/\theta)$ $(x > 0, \theta > 0)$. Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for the parameter θ.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a Bernoulli(p) distribution, that is, $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$ where $0 < p < 1$ is assumed.

(a) Derive the likelihood function of the parameter p.

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(c) Show that the maximum likelihood estimator attains the Cramér-Rao Lower Bound.

3. Let $X_1, ..., X_n$ be a random sample from a distribution whose probability density function is

$$f(x;\theta) = 2\theta x \exp\left(-\theta x^2\right), \quad x > 0, \quad \theta > 0.$$

Derive the likelihood ratio test for testing H_0: $\theta = \theta_0$ versus H_1: $\theta = \theta_0$.

4. Let $X_1, ..., X_n$ be a random sample from a $n(\mu, 1)$ distribution (normal distribution with expectation μ and variance 1). Derive a confidence interval for the expectation μ.

Note: If the random variable X has a $n\left(\mu, \sigma^2\right)$ distribution its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

Statistical inference - Exam June 12, 2008 (Casella&Berger)

1. (a) Explain what is meant by a sufficient statistic (definition, meaning, interpretation).

(b) Suppose that $X_1, ..., X_n$ is a random sample from an exponential distribution with probability density function $f(x) = (1/\theta)\exp(-x/\theta)$ $(x > 0, \theta > 0)$. Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for the parameter θ.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a Bernoulli(p) distribution, that is, $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$ where $0 < p < 1$ is assumed.
(a) Derive the likelihood function of the parameter p.
(b) Derive the maximum likelihood estimator of the parameter p.
(c) Show that the maximum likelihood estimator attains the Cramér-Rao Lower Bound.

3. Let $X_1, ..., X_n$ be a random sample from a distribution whose probability density function is

$$f(x;\theta) = \theta^{-2} x \exp(-x/\theta), \quad x > 0, \quad \theta > 0.$$

Derive the likelihood ratio (LR) test, Wald test and score (LM/Rao) test for testing H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$. (Note: Integation yields $E(X_i) = 2\theta$ and $var(X_i) = 2\theta^2$.)

4. Let $X_1, ..., X_n$ be a random sample from a $n(\mu, 1)$ distribution (normal distribution with expectation μ and variance 1). Derive a confidence interval for the expectation μ.

Note: If the random variable X has a $n(\mu, \sigma^2)$ distribution its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

Statistical inference - Exam August 14th, 2008 (Casella&Berger)

1. (a) Explain what is meant by a sufficient statistic (definition, meaning, interpretation).

(b) Suppose that $X_1, ..., X_n$ is a random sample from an exponential distribution with probability density function $f(x) = (1/\theta)\exp(-x/\theta)$ $(x > 0, \theta > 0)$. Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for the parameter θ.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a Bernoulli(p) distribution, that is, $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$ where $0 < p < 1$ is assumed.
(a) Derive the likelihood function of the parameter p.
(b) Derive the maximum likelihood estimator of the parameter p.
(c) Show that the maximum likelihood estimator attains the Cramér-Rao Lower Bound.

3. Let $X_1, ..., X_n$ be a random sample from a distribution whose probability density function is

$$f(x; \theta) = \theta^{-2} x \exp(-x/\theta), \quad x > 0, \quad \theta > 0.$$

Derive the likelihood ratio (LR) test, Wald test and score (LM/Rao) test for testing H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$. (Note: Integation yields $E(X_i) = 2\theta$ and $var(X_i) = 2\theta^2$.)

4. Let $X_1, ..., X_n$ be a random sample from a $n(\mu, 1)$ distribution (normal distribution with expectation μ and variance 1). Derive a confidence interval for the expectation μ.

Note: If the random variable X has a $n(\mu, \sigma^2)$ distribution its density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

STATISTICAL INFERENCE, 10 credits (intermediate studies). Literature: Pekka Nieminen and Pentti Saikkonen: Tilastollisen päättelyn kurssi. Lecturer: University Lecturer Pekka Pere.

Renewal exam (whole course) 2.4.2009

The questions are all equally worthy. In the first two questions it is assumed that the parameter is univariate. Throughout assume that the regularity conditions (as explained in the lecture notes) apply. Please return the question sheet.

1. Let the model parameter $\theta \in \Omega$ be univariate. Carefully define and explain with words and formulae

a) the maximum likelihood (ML) $\hat{\theta}$ estimator and the asymptotic ($n \longrightarrow \infty$) properties of it (e.g. the asymptotic distribution of it).
b) the invariance property of ML estimators, or estimation of $g(\theta)$ (using the notation of the lecture notes) by the ML method.
c) Give the formula for the information inequality (informaatioepäyhtälö), and explain in words the meaning of it. How does it relate to the properties of the ML estimator?
d) Define and explain in words the meaning of efficiency of the estimator T. What kind of an estimator fully efficient? Does such an estimator always exist? When at least is an ML estimator efficient?

2. Let us denote by T an estimator of a univariate parameter $\theta \in \Omega$. Carefully define and explain with words and formulae

a) unbiasedness and mean squared error of an estimator T. How does the mean squared error relate to the variance of an estimator? Derive the associated formula.
b) a sufficient statistic and the factorization criterion, and why it is useful.
c) the Neyman–Pearson lemma.

d) confidence interval (the frequentist interpretation of it).

3. Let $Y_1, \ldots, Y_n \sim \mathsf{N}(\mu_0, \sigma^2) \perp\!\!\!\perp$ (independently and identically normally distributed). Here the mean $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$.

a) What is the joint density function of the observations (n of them)? Give the formula, and explain it carefully.
b) What is the associated likelihood function? Give the formula, and explain it carefully.
c) Derive the expected (Fisher) information $i(\mu)$ for the mean μ. Interprete the formula you have derived in words. How des expected information relate — or does it relate at all — to the the variance of the ML estimator of μ?
d) Formulate the Rao statistic for testing the null hypothesis $\mu = 0$.

Auxiliary results:

- The density function of a Normally distributed random variable $Y \sim \mathsf{N}(\mu, \sigma^2)$ is $(2\pi\sigma^2)^{-1/2} \exp[-(y_i - \mu)^2/2\sigma^2]$ (with obvious notation).
- $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = (n-1) + (n-2) + \cdots + 1 = n(n-1)/2.$

STATISTICAL INFERENCE, 10 credits (intermediate studies). Literature: Pekka Nieminen and Pentti Saikkonen: Tilastollisen päättelyn kurssi. Lecturer: University Lecturer Pekka Pere.

General exam 12.5.2009

The questions are all equally worthy. Throughout assume that the regularity conditions (as explained in the lecture notes) apply. Please return the question sheet and the tables.

1. Let us denote by T an estimator of (a univariate) parameter $\theta \in \Omega$. Carefully define and explain with words and formulae
a) unbiasedness (harhattomuus) of estimator T.
b) mean squared error (keskineliövirhe) of estimator T. How does it relate to the variance of an estimator? Derive the associated formula.
c) consistency of estimator T.
d) a sufficient statistic and the factorization criterion.

2. Let $Y_1, \ldots, Y_n \sim \mathsf{N}(\mu_0, \sigma^2) \perp\!\!\!\perp$ (independently and identically normally distributed). The mean (μ_0) is known in advance but the standard deviation ($\sigma > 0$) is unknown.
a) What is the joint density function of the observations (n of them)? Give the formula, and explain carefully.
b) Derive the ML estimator $\hat{\sigma}$ for the standard deviation σ. NB: The ML estimator for the standard deviation σ instead of the variance σ^2 is asked for. Check that you have found the (local) maximum of the log-likelihood.
c) Derive expected (Fisher) information $i(\sigma)$ for the standard deviation σ. (Hint: Calculate $i(\sigma)$ by means of the formula which is easier in general according to the lecture notes. If you have time, check your result with the other formula which will yield a much more laborous derivation.)
d) What is the approximative distribution of ML estimator $\hat{\sigma}$ for large samples? Alternatively, give the exact asymptotic distribution for suitably standardized ML estimator $\hat{\sigma}$ and lay out the suitable standafrdization. Explain and interpret the formula you give.

3. Let the model parameter $\theta \in \Omega$ be univariate. Carefully define and explain with words and formulae
a) observed significance level of a test (p).
b) likelihood ratio, Wald and Rao test statistics, and explain how they are used (the reference distributions and the justification for them).
c) confidence interval (the frequentist interpretation of it).
d) the justification for the much used confidence interval $\hat{\theta} \pm 2 \times s.e.(\hat{\theta})$ (using the notation of the lecture notes).

4. Exponentially, independently and identically distributed random variables $Y_1, \ldots, Y_n \sim \mathsf{Exp}(1/\mu)$ are inspected. The null hypothesis is $H_0 : \mu = \mu_0$, where $\mu_0 > 0$.
a) Derive the formulae for the likelihood ratio, Wald and Rao test statistics.
b) The sample size is $n = 50$, and the sample mean is $\bar{y} = 800$. Test with the above mentioned test statistics, if the null hypothesis $H_0 : \mu = 1000$ is rejected or not at significance level 0,05 (two-sided test).

Auxiliary results (not all are necessarily needed):

- The density function of a Normally distributed random variable $Y \sim \mathsf{N}(\mu, \sigma^2)$ is $(2\pi\sigma^2)^{-1/2} \exp[-(y_i - \mu)^2/2\sigma^2]$ (with obvious notation).
- The fourth central moment $\mathsf{E}(Y - \mu)^4$ of a random variable Y, which follows the Normal distribution $\mathsf{N}(\mu, \sigma^2)$, is $3\sigma^4$.
- $(\sum_{i=1}^n y_i)^2 = (\sum_{i=1}^n \sum_{j=i}^n y_i y_j)^2 = \sum_{i=1}^n y_i^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^n y_i y_j.$
- $\sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = (n-1) + (n-2) + \cdots + 1 = n(n-1)/2.$
- The density function of an exponentially distributed ($\mathsf{Exp}(1/\mu)$) random variable Y is $(1/\mu)\exp(-y/\mu)$ (p. 3 of the lecture notes).
- If the random variables are exponentially distributed ($\mathsf{Exp}(1/\mu)$) then the ML estimator for μ is $\bar{y}$, where $\bar{y} = n^{-1}\sum_{i=1}^n y_i$ (p. 16 of the lecture notes) and Fisher informaation $i(\mu) = n/\mu^2$ (p. 70 of the lecture notes).
- Accompanying tables.

STATISTICAL INFERENCE, 10 credits (intermediate studies). Literature: Pekka Nieminen and Pentti Saikkonen: Tilastollisen päättelyn kurssi. Examiner: University Lecturer Pekka Pere.

General examination 11.6.2009

The questions are all equally worthy. In the first two questions it is assumed that the parameter is univariate. Throughout assume that the usual regularity conditions apply. Please return the question sheet.

1. Let the model parameter $\theta \in \Omega$ be univariate. Carefully define and explain with words and formulae
a) the maximum likelihood (ML) estimator $\hat{\theta}$ and the asymptotic ($n \longrightarrow \infty$) properties of it (e.g. the asymptotic distribution of it).
b) the invariance property of ML estimators, or estimation of $g(\theta)$ (using the notation of the lecture notes) by the ML method.
c) Give the formula for the information inequality and explain in words the meaning of it. How does it relate to the properties of the ML estimator?
d) Define and explain in words the meaning of efficiency of the estimator T. What kind of an estimator fully efficient? Does such an estimator always exist? When at least is an ML estimator efficient?

2. Let us denote by T an estimator of a univariate parameter $\theta \in \Omega$. Carefully define and explain with words and formulae
a) unbiasedness and mean squared error of an estimator T. How does the mean squared error relate to the variance of an estimator? Derive the associated formula.
b) a sufficient statistic and the factorization criterion, and why it is useful.
c) the Neyman–Pearson lemma.
d) confidence interval (the frequentist interpretation of it).

3. Let $Y_1, \ldots, Y_n \sim \mathsf{N}(\mu_0, \sigma^2) \perp\!\!\!\perp$ (independently and identically normally distributed). Here the mean $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$.
a) What is the joint density function of the observations (n of them)? Give the formula, and explain it carefully.
b) What is the associated likelihood function? Give the formula, and explain it carefully.
c) Derive the expected (Fisher) information $i(\mu)$ for the mean μ. Interprete the formula you have derived in words. How des expected information relate — or does it relate at all — to the the variance of the ML estimator of μ?
d) Formulate the score or Lagrange multiplier statistic for testing the null hypothesis $\mu = 0$.

Auxiliary result: The density function of a Normally distributed random variable $Y \sim \mathsf{N}(\mu, \sigma^2)$ is $(2\pi\sigma^2)^{-1/2} \exp[-(y_i - \mu)^2/2\sigma^2]$ (with obvious notation).

It is likely that the exam results will be given not as quickly as usual because of the summer. Enjoy statistics and the summer in the meanwhile!