Statistical inference — Final exam 16. 11. 2006

1. a) What is the definition of a sufficient statistic (in terms of conditional distributions)? Also write a sentence or two explaining the practical interpretation of sufficiency.
   b) Let $X_1, \ldots, X_n$ be iid exponential(λ). Find a real-valued sufficient statistic for the parameter λ.

2. Consider a sequence of n independent Bernoulli trials, where the success probability of each trial is an unknown θ. The random variable X_i takes the value 1 if the i:th trial is a success, and the value 0 if it is a failure, so that $P(X_i = 1) = \theta$ and $P(X_i = 0) = 1 - \theta$. Derive the likelihood function corresponding to observed data $x_1, \ldots, x_n$, and calculate the maximum likelihood estimate of θ. (Justify your conclusions carefully.)

3. Let X_1, X_2, X_3 be three independent observations from the exponential(λ) distribution. Define

$$S = \tfrac{1}{4}(2X_1 + X_2 + X_3), \qquad T = \tfrac{1}{3}(X_1 + X_2 + X_3).$$

   Verify that S and T are unbiased estimators of λ. Why is T better? Prove that T is, in fact, the best unbiased estimator of λ.

4. A random variable X takes values in the set $\{0, 1, 2, 3, 4\}$, and its distribution depends on a parameter θ, which has three possible values: 1, 2 and 3. The probability mass function $f_X(x \mid \theta)$ of X is as follows:

x	0	1	2	3	4
$f_X(x \mid 1)$	.03	.02	.05	.80	.10
$f_X(x \mid 2)$	.05	.05	.80	.10	.00
$f_X(x \mid 3)$	.60	.38	.02	.00	.00

   One is willing to test $H_0\colon \theta = 1$ versus $H_1\colon \theta = 3$. What is the rejection region (or criticial region) of the uniformly most powerful (UMP) level 0.05 test? Also determine the power function of this test.

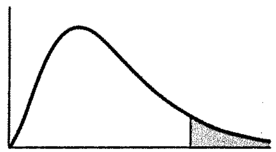
5. a) Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random sample whose distribution depends on a real-valued parameter θ. What is the definition of an interval estimator of θ with confidence coefficient $1 - \alpha$ (also known as a $1 - \alpha$ confidence interval)?
   b) Let $X_1, \ldots, X_{10}$ be iid $\mathrm{n}(\mu, \sigma^2)$. It is well known from probability theory that if $S^2 = \sum_{i=1}^{10}(X_i - \overline{X})^2/9$, then $9S^2/\sigma^2 \sim \chi^2_9$ (chi squared distribution with 9 degrees of freedom). Use this fact to derive a 90 % confidence interval for the standard deviation σ, given that the observed value of S is $s = 4.5$. (See overleaf for a table of cutoff points of χ^2 distributions.)

You may use the following facts if you need:

If $X \sim$ exponential(λ), then the pdf of X is $f(x \mid \lambda) = e^{-x/\lambda}/\lambda$, $x > 0$, and $\mathrm{E}X = \lambda$, $\mathrm{Var}\,X = \lambda^2$.

If $X \sim \mathrm{n}(\mu, \sigma^2)$, then the pdf of X is $f(x \mid \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\{-(x-\mu)^2/2\sigma^2\}$, and $\mathrm{E}X = \mu$, $\mathrm{Var}\,X = \sigma^2$.

TABLE 3 Cutoff points for the chi squared distribution, right-hand tail probabilities



df	.9995	.999	.9975	.995	.990	.975	.950	.900	.750	.500
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.10	0.45
2	0.00	0.00	0.01	0.01	0.02	0.05	0.10	0.21	0.58	1.39
3	0.02	0.02	0.04	0.07	0.11	0.22	0.35	0.58	1.21	2.37
4	0.06	0.09	0.14	0.21	0.30	0.48	0.71	1.06	1.92	3.36
5	0.16	0.21	0.31	0.41	0.55	0.83	1.15	1.61	2.67	4.35
6	0.30	0.38	0.53	0.68	0.87	1.24	1.64	2.20	3.45	5.35
7	0.48	0.60	0.79	0.99	1.24	1.69	2.17	2.83	4.25	6.35
8	0.71	0.86	1.10	1.34	1.65	2.18	2.73	3.49	5.07	7.34
9	0.97	1.15	1.45	1.73	2.09	2.70	3.33	4.17	5.90	8.34
10	1.26	1.48	1.83	2.16	2.56	3.25	3.94	4.87	6.74	9.34
11	1.59	1.83	2.23	2.60	3.05	3.82	4.57	5.58	7.58	10.34
12	1.93	2.21	2.66	3.07	3.57	4.40	5.23	6.30	8.44	11.34
13	2.31	2.62	3.11	3.57	4.11	5.01	5.89	7.04	9.30	12.34
14	2.70	3.04	3.58	4.07	4.66	5.63	6.57	7.79	10.17	13.34
15	3.11	3.48	4.07	4.60	5.23	6.26	7.26	8.55	11.04	14.34
16	3.54	3.94	4.57	5.14	5.81	6.91	7.96	9.31	11.91	15.34
17	3.98	4.42	5.09	5.70	6.41	7.56	8.67	10.09	12.79	16.34
18	4.44	4.90	5.62	6.26	7.01	8.23	9.39	10.86	13.68	17.34
19	4.91	5.41	6.17	6.84	7.63	8.91	10.12	11.65	14.56	18.34
20	5.40	5.92	6.72	7.43	8.26	9.59	10.85	12.44	15.45	19.34
21	5.90	6.45	7.29	8.03	8.90	10.28	11.59	13.24	16.34	20.34
22	6.40	6.98	7.86	8.64	9.54	10.98	12.34	14.04	17.24	21.34
23	6.92	7.53	8.45	9.26	10.20	11.69	13.09	14.85	18.14	22.34
24	7.45	8.08	9.04	9.89	10.86	12.40	13.85	15.66	19.04	23.34
25	7.99	8.65	9.65	10.52	11.52	13.12	14.61	16.47	19.94	24.34
26	8.54	9.22	10.26	11.16	12.20	13.84	15.38	17.29	20.84	25.34
27	9.09	9.80	10.87	11.81	12.88	14.57	16.15	18.11	21.75	26.34
28	9.66	10.39	11.50	12.46	13.56	15.31	16.93	18.94	22.66	27.34
29	10.23	10.99	12.13	13.12	14.26	16.05	17.71	19.77	23.57	28.34
30	10.80	11.59	12.76	13.79	14.95	16.79	18.49	20.60	24.48	29.34
40	16.91	17.92	19.42	20.71	22.16	24.43	26.51	29.05	33.66	39.34
50	23.46	24.67	26.46	27.99	29.71	32.36	34.76	37.69	42.94	49.33
60	30.34	31.74	33.79	35.53	37.48	40.48	43.19	46.46	52.29	59.33
80	44.79	46.52	49.04	51.17	53.54	57.15	60.39	64.28	71.14	79.33
100	59.90	61.92	64.86	67.33	70.06	74.22	77.93	82.36	90.13	99.33

TABLE 3 (continued)

df	.250	.200	.150	.100	.050	.025	.020	.010	.005	.0025	.001	.0005
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51	22.11
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80	49.01
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48	57.86
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.09
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2