STATISTICAL INFERENCE, 10 credits (intermediate studies). Literature: Pekka Nieminen and Pentti Saikkonen: Tilastollisen päättelyn kurssi. Lecturer: University Lecturer Pekka Pere.

1. intermediate exam 22.1.2009 (renewal)

The three questions are worth six points each. Throughout assume that the regularity conditions (as explained in the lecture notes) apply. Please return the question sheet (thank you!).

1. Let the model parameter $\theta \in \Omega$ be univariate. Carefully define and explain
a) the maximum likelihood (ML) $\hat{\theta}$ estimator (suurimman uskottavuuden (SU) estimaattori) and the asymptotic ($n \longrightarrow \infty$) properties of it (e.g. the asymptotic distribution of it).
b) bias of an estimator (T) of θ. Is the ML estimator of θ always unbiased? Explain carefully.
c) the invariance property of ML estimators, or estimation of $g(\theta)$ (using the notation of the lecture notes) by the ML method.
d) observed (havaittu) and expected or Fisher information (odotettu informaatio). How do they differ in meaning? How does expected information relate to the asymptotic properties of the ML estimator?

2. Let T be an unbiased estimator of a univariate parameter $\theta \in \Omega$.
a) Give the formula for the information inequality (informaatioepäyhtälö), and explain in words the meaning of it. How does it relate to the properties of the ML estimator?
b) Define and explain in words the meaning of efficiency (tehokkuus) of the estimator T. What kind of an estimator fully efficient (täystehokas)? Does such an estimator always exist? You gain an extra point if you answer justifiably to this question: When at least is an ML estimator efficient?

3. Let $Y_1, \ldots, Y_n \sim \mathsf{N}(\mu_0, \sigma^2) \perp\!\!\!\perp$ (independently and identically normally distributed). The mean (μ_0) is known in advance but the standard deviation ($\sigma > 0$) is unknown.
a) What is the joint density function of the observations (n of them)? Give the formula, and explain carefully.
b) Derive the ML estimator $\hat{\sigma}$ for the standard deviation σ. NB: The ML estimator for the standard deviation σ instead of the variance σ^2 is asked for. Check that you have found the (local) maximum of the log-likelihood.
c) Derive expected (Fisher) information $i(\sigma)$ for the standard deviation σ. (Hint: Calculate $\imath(\sigma)$ by means of the formula which is easier in general according to the lecture notes. If you have time, check your result with the other formula which will yield a much more laborous derivation.)

d) What is the approximative distribution of ML estimator $\hat{\sigma}$ for large samples? Alternatively, give the exact asymptotic distribution for suitably standardized ML estimator $\hat{\sigma}$ and lay out the suitable standafrdization. Explain and interpret the formula you give.

Auxiliary results:

- The density function of a Normally distributed random variable $Y \sim \mathsf{N}(\mu, \sigma^2)$ is $(2\pi\sigma^2)^{-1/2} \exp[-(y_i - \mu)^2/2\sigma^2]$ (with obvious notation).
- The fourth central moment $\mathsf{E}(Y - \mu)^4$ of a random variable Y, which follows the Normal distribution $\mathsf{N}(\mu, \sigma^2)$, is $3\sigma^4$.
- $(\sum_{i=1}^{n} y_i)^2 = (\sum_{i=1}^{n} \sum_{j=i}^{n} y_i y_j)^2 = \sum_{i=1}^{n} y_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_i y_j$.
- $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = (n-1) + (n-2) + \cdots + 1 = n(n-1)/2$.

You do not necessarily need the last three results in the exam.