Set Theory Final Exam

20.1.2004

1. Describe how the hierarchy of V_α's are defined. Is the axiom of choice used?

2. If η is the order type of the rationals and λ is the order type of the reals, prove that $\bar{1}+\bar{\eta} \neq \bar{\eta}+\bar{1}$ and $\lambda+\lambda\neq\lambda$.

3. Prove that for all sets x, $x \notin x$.

4. Prove the Cantor-Schröder-Bernstein theorem: For any sets A and B, if $A \preceq B$ and $B \preceq A$ then $A \approx B$.

5. Compute $\aleph_0^{2^{\aleph_0}} \cdot 2^{\aleph_0}$

6. Prove: $\aleph_0 \leq \kappa$ for any infinite cardinal κ.

7. Define a normal operation and give an example.

Set Theory Exam

J.K.

March 2004

1. Prove that $(0,1) \approx \mathbb{R}$. ("$\approx$" means, is equinumerous.) You may use the Cantor-Schröder-Bernstein theorem.

2. Define the ranked hierarchy by transfinite recursion, where the ranked hierarchy are the V_α's.

3. Compute: $2^{\aleph_0} \cdot \aleph_0^{\aleph_0}$, meaning simplify.

4. Prove that the well-ordering axiom (every set can be well-ordered) implies Zorn's Lemma.

5. If α, β, γ are ordinals, prove $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

6. If κ, λ, μ are cardinal numbers, prove $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$

Elements of Set Theory

Answer the following questions:

April 12, 2005

1. Prove the Schroder-Bernstein theorem: for A and B sets, if f is a one to one mapping of A into B, and g is a one to one mapping of B into A, then A and B are equinumerous.
2. Define the natural numbers as well as the multiplication on the natural numbers, $\cdot_\omega$. Prove that this multiplication is associative.
3. Define the real numbers and prove that for any real number x, there is some rational q in x such that $p + q \notin x$, where p is a positive rational number.
4. Prove that the axiom of choice implies Zorn's lemma: every partially ordered set which is closed under unions of chains has a maximal element.
5. Compute $2^{\aleph_1} \cdot \aleph_1^{\aleph_0}$.

Department of Mathematics and Statistics
Advanced Set Theory

Final Exam 18.5.2006/Huuskonen

1. Show that the following conditions are equivalent:
   a) $\mathbf{V} = \mathbf{OD}$.
   b) $\mathbf{V} = \mathbf{HOD}$.
   c) $\mathbf{OD}$ is transitive.
   d) The Axiom of Extensionality is true in $\mathbf{OD}$.
2. Let A be a set and $B \subseteq A$ finite. Show that $B \in \mathrm{Df}(A, 1)$.
3. Let $x, y \in \mathbf{L}$. Calculate $\rho(\{x, y\})$ and $\rho(\langle x, y\rangle)$, given $\rho(x)$ and $\rho(y)$. Here $\rho(x)$ denotes the smallest index α such that $x \in L(\alpha + 1)$.
4. Let M be a countable transitive model of ZFC, $\langle P, \Vdash, 1\rangle \in M$ a notion of forcing, and φ and ψ sentences in the language of set theory. Show, by using the definition of forcing, that $p \Vdash \varphi \to \psi$ iff for all conditions $q \Vdash p$ such that $q \Vdash \varphi$ there is $r \Vdash q$ such that $r \Vdash \psi$.
5. Let M be a countable transitive model of ZFC. Using forcing, construct an extension of M satisfying ZFC and $2^{\aleph_0} > \aleph_\omega$. All general properties of forcing proved on the course may be assumed known.

Set theory Exam.

11.12.06

1. Prove that for any infinite cardinal κ, $\kappa \cdot \kappa = \kappa$.

2. Prove AC is equivalent to cardinal comparability, where AC is: $\forall u \, \exists f \, (u \neq \emptyset \Rightarrow \forall x \, [x \in u \Rightarrow f(x) \in x]))$ and cardinal comp: $\forall \alpha, \beta$ either $\aleph_\alpha \preceq \aleph_\beta$ or conversely.

3, 4: Define $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: natural numbers, integers, rationals, reals. And the operations $+_{\mathbb{N}}, \cdot_{\mathbb{N}}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, \ldots$ etc. Show that each is embedded as a substructure of the next, ie show that $f_{\mathbb{N}}, f_{\mathbb{Z}}, f_{\mathbb{Q}}, f_{\mathbb{R}}$ all 1-1 and operation preserving s.t. $f_{\mathbb{N}} : \mathbb{N} \overset{1\text{-}1}{\hookrightarrow} \mathbb{Z}$, $f_{\mathbb{Z}} : \mathbb{Z} \overset{1\text{-}1}{\to} \mathbb{Q}$, $f_{\mathbb{Q}} : \mathbb{Q} \overset{1\text{-}1}{\hookrightarrow} \mathbb{R}$.

5. Define the ordinals by transfinite recursion.

Set Theory Exam.

20.12.06

1. Prove that for any infinite cardinal κ, $\kappa \cdot \kappa = \kappa$.

2. Prove AC is equivalent to cardinal comparability, where AC is: $\forall u \exists f$ $(u \neq \emptyset \Rightarrow \forall x [x \in u \Rightarrow f(x) \in x]))$, $\varepsilon x \neq \emptyset$ and cardinal comp: $\forall \alpha, \beta$ either $\aleph_\alpha \preceq \aleph_\beta$ or conversely.

3, 4: Define $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: natural numbers, integers, rationals, reals. And the operations $+_{\mathbb{N}}, \cdot_{\mathbb{N}}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, \ldots$ etc. Show that each is embedded as a substructure of the next, ie show that $\exists f_{\mathbb{N}}, f_{\mathbb{Z}}, f_{\mathbb{Q}}, f_{\mathbb{R}}$ all 1-1 and operation preserving s.t. $f_{\mathbb{N}} : \mathbb{N} \overset{1\text{-}1}{\hookrightarrow} \mathbb{Z}$, $f_{\mathbb{Z}} : \mathbb{Z} \overset{1\text{-}1}{\to} \mathbb{Q}$, $f_{\mathbb{Q}} : \mathbb{Q} \overset{1\text{-}1}{\hookrightarrow} \mathbb{R}$.

5. Define the ordinals by transfinite recursion.