Matematiikan ja tilastotieteen laitos
Real Analysis I
Final exam
9.8.2007

1. Define absolute continuity of measures. Give examples of measures μ and ν such that a) μ is absolutely continuous with respect to ν but ν is not absolutely continuous with respect to μ, b) μ is not absolutely continuous with respect to ν and ν is not absolutely continuous with respect to μ.

2. Let (X, Γ, μ) be a measure space and $1 \leq p < q < r \leq \infty$. Show that if $f \in L^q(\mu)$, then there exists $g \in L^p(\mu)$ and $h \in L^r(\mu)$ such that $f = g + h$.

3. Define convolution. Show that if $f, g \in L^1(\mathbb{R}^n)$ and g is uniformly continuous, then $f * g$ is continuous.

4. Show that if μ is a measure in the σ-algebra of Borel sets of $\mathbb{R}^n$ and $\mu(B(x,r)) \leq m(B(x,r))$ for all $x \in \mathbb{R}^n$ and $r > 0$, then $\mu(U) \leq m(U)$ for all open sets $U \subset \mathbb{R}^n$.

5. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $1 < p < \infty$ and $f' \in L^p([a,b])$. Show that there is $C \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha},$$

where $\alpha = 1 - 1/p$.