Department of Mathematics
Real Analysis I
Final exam
August 10, 2004

1. Let (X, Γ, μ) be a measure space, $\mu(X) = 1$, and $f\colon X \to]0, \infty[$ an integrable function. Show that
$$\int_X \log f(x)\, d\mu(x) \leq \log\left(\int_X f(x)\, d\mu(x)\right).$$
[Hint: Apply an inequality $\log t \leq t - 1$ with $t = f(x)/\|f\|_1$.]

2. Formulate and prove Minkowski's inequality for L^p-spaces, $1 < p < \infty$. Hölder's inequality is assumed.

3. Prove that L^p-spaces are complete for $1 \leq p < \infty$. The following result is assumed: If (f_j) is a Cauchy-sequence in L^p, then there exists a subsequence (f_{j_k}) converging almost everywhere.

4. Let $f\colon [a, b] \to \mathbb{R}$ be increasing. Show (by applying Lebesgue's differentiability theorem for monotone functions) that the derivative f' is integrable on the interval $[a, b]$ and that
$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

5. Prove that an absolutely continuous function $f\colon [a, b] \to \mathbb{R}$ is of bounded variation.

Department of Mathematics and Statistics
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24.5.2005

1. Let (X, Γ, μ) be a measure space with $\mu(X) < \infty$. Show that for $1 \leq q < p \leq \infty$ it holds
$$L^p(\mu) \subset L^q(\mu).$$
Moreover, show that the embedding $I : L^p(\mu) \to L^q(\mu)$, $I(f) = f$ is continuous.

2. Let $f, g, h \in L^1(\mathbb{R}^n)$. Show that
(a) $f * g = g * f$,
(b) $f * (g * h) = (f * g) * h$,
(c) $f * (g + h) = f * g + f * h$.

3. Let $f \in L^1(\mathbb{R}^n)$. Show that $Mf(x) < +\infty$ for almost every $x \in \mathbb{R}^n$.

4. Let $I = (0,1) \times (0,1)$ be the open unite square in the plane $\mathbb{R}^2$. Let
$$f(x) = 1, \text{ for } x \in I \text{ and } f(x) = 0, \text{ for } X \in \mathbb{R}^2 \setminus \overline{I}.$$
Show that one can define f on the boundary ∂I:n of I s.t.
$$\lim_{r \to 0} \fint_{B(x,r)} f(y)dy = f(x)$$
holds for every $x \in \mathbb{R}^2$.

5. Let
$$f(x) = \begin{cases} x^{1/2} \log x, & \text{for } x \in (0,1] \\ 0, & \text{for } x = 0. \end{cases}$$
Is f absolutely continuous on $[0,1]$. Hint: Calculate the derivative of f.

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1. Prove Minkowski's inequality for L^p-spaces, $1 < p < \infty$, by using Hölder's inequality.

2. Formulate Egorov's theorem and prove by using it the following claim: Suppose that $A \subset \mathbb{R}^n$ is measurable, $m(A) < \infty$, $1 \leq p < \infty$, $f, f_1, f_2, \ldots \in L^p(A)$, and that $f_j \to f$ almost everywhere. Then, for every $\varepsilon > 0$, there exists a compact set $F \subset A$ such that

$$\int_{A \setminus F} |f|^p dm < \varepsilon$$

and $f_j|F \to f|F$ uniformly.

3. (a) Define the Hardy-Littlewood maximal function Mf for a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.
(b) Prove that $Mf\colon \mathbb{R}^n \to [0, +\infty]$ is a measurable function.

4. Let $I =]0,1[\times]0,1[\subset \mathbb{R}^2$ and

$$f(x) = \begin{cases} 1, & \text{if } x \in I; \\ 0, & \text{if } x \in \mathbb{R}^2 \setminus \bar{I}. \end{cases}$$

Prove that f can be defined on the boundary ∂I of I such that

$$\lim_{r \to 0+} \frac{1}{B(x,r)} \int_{B(x,r)} f(y)dy = f(x)$$

holds for every $x \in \mathbb{R}^2$.

5. Prove that an absolutely continuous function $f\colon [a,b] \to \mathbb{R}$ is of bounded variation.