UH / Department of Mathematics and Statistics **Exam**
Probability theory **4.3.2008**

Note: You have **4 hours** to write the exam.

1. (a) Formulate and prove the first Borel-Cantelli lemma.
   (b) Formulate and prove the second Borel-Cantelli lemma.

2. For random variables X and Y we define

$$d(X,Y) := E\Big(\frac{|X-Y|}{1+|X-Y|}\Big)$$

   Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables

   (a) Show that if $X_n \xrightarrow{P} Y$ (in the sense of convegence in probability), then $d(X_n, Y) \to 0$.
   (b) Show that if $d(X_n, Y) \to 0$, then $X_n \xrightarrow{P} Y$ (in the sense of convegence in probability).

3. Let (X_n) a sequence of independent and identically distributed random variables with values in $(0,\infty)$ such that $E(|\log X_1|) < \infty$.

   Show that P-almost surely,

$$\lim_{n\to\infty} \Big(\prod_{k=1}^{n} X_k\Big)^{1/n} = \exp(\ E[\ \log(X_1)\ ]\ )$$

4. (a) Sketch the proof of Fubini theorem (about changing the order of integration on product space).
   (b) Let $X \geq 0$ be a non-negative real valued random variable. Show that

$$E(X) = \int_0^\infty P(X > t)dt$$

   (c) Is it true that we have also

$$E(X) \stackrel{?}{=} \int_0^\infty P(X \geq t)dt\ ,$$

   even when the distribution function $F(t) = P(X \leq t)$ is not continuous?

5. Let $(X_1, \ldots, X_n)$ independent and identically distributed random variables, with $X_1 \in L^1(P)$, and let

$$S_n = (X_1 + X_2 + \ldots + X_n)$$

   (a) Use the Kolmogorov definition of conditional expectation to show the symmetry

$$E(X_k|\sigma(S_n)) = E(X_1|\sigma(S_n)) \text{ for } k = 1, \ldots, n$$

   (b) Using symmetry, compute $E(X_1|\sigma(S_n))$.

Probability theory , Exam , 19.05.09

NOTE: THE EXAM IS DIVIDED IN TWO PARTS, THOSE WHO HAVE PASSED THE FIRST MIDDLE TERM EXAM DURING THIS SEMESTER, ARE ASKED TO SOLVE PART II ONLY (PROBLEMS 4-5), AND IN THAT CASE DO NOT NEED TO ANSWER THE QUESTIONS FROM PART I (PROBLEMS 1-3). THE EXAM LASTS 4 HOURS FOR EVERYBODY. IT IS NOT ALLOWED TO USE LECTURE MATERIAL OR LITERATURE. MATHEMATICAL TABLES AND POCKET CALCULATOR ARE ALLOWED BUT AREN'T NEEDED.

1 PART I

Problem 1 On a probability space $(\Omega, \mathcal{F}, P)$, consider a random variable $X(\omega) \geq 0$ P-almost surely. Show that

$$E_P(X) = \int_0^\infty P(X > t)dt$$

Hint: use Fubini, checking whether the assumption of Fubini theorem hold .

Problem 2 On a probability space $(\Omega, \mathcal{F}, P)$, let $X_1(\omega), X_2(\omega), \dots, X_n, \dots$ a sequence of independent and identically distributed λ-exponential random variables for some parameter $\lambda > 0$, that is

$$P(X_1 \leq t) = \mathbf{1}(t > 0)\Big\{1 - \exp(-\lambda t)\Big\}$$

2.i) Show that

$$D_n(\omega) := \min\{X_k(\omega) : 1 \leq k \leq n\}$$

is a random variable, and that its distribution is also exponential.

2.ii) Show that

$$U_n(\omega) := \max\{X_k(\omega) : 1 \leq k \leq n\}$$

is a random variable and compute its density. (Hint: compute first the cumulative distribution function).

2.iii) Compute the expectation

$$E_P(U_n) := E_P\Big(\max_{1\le k\le n}\{X_k(\omega)\}\Big)$$

Hint: use the result from Problem 1.

Problem 3 On a probability space $(\Omega, \mathcal{F}, P)$, let $X_1(\omega), X_2(\omega), \ldots, X_n(\omega), \ldots$ a sequence of independent random variables (which are **not** identically distributed), such that

$$P\Big(X_n = n^2 - 1\Big) = \Big\{1 - P\Big(X_n = -1\Big)\Big\} = n^{-2}\ ,$$

and let $S_n := (X_1 + X_2 + \cdots + X_n)$.

3.i) Show that for P almost all ω

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = -1\ ,$$

while for all n

$$E_P\Big(\frac{S_n}{n}\Big) = 0.$$

Hint: compute first

$$P\Big(\omega : X_n(\omega) \neq -1 \text{ for infinitely many } n\ \Big)$$

3.ii) Explain why in this case Fatou lemma applies for the sequence $\{n^{-1}S_n(\omega)\}$, but reverse Fatou lemma does not apply.

2 PART II

Problem 4 On a probability space $(\Omega, \mathcal{F}, P)$,

Let $(X_1(\omega), \dots, X_n(\omega), \dots)$ independent random variables with respective distributions Poisson(λ_i), $i = 1, \dots, n$. for parameter values $\lambda_i > 0$, that is

$$P(X_i = k) = \exp(-\lambda_i)\frac{\lambda_i^k}{k!}$$

4.i) Compute the characteristic function of the Poisson distribution:

$$\varphi(\theta) = E_P(\exp(i\theta X_1))$$

4.ii) Show that the convolution

$$S_n(\omega) = (X_1(\omega) + X_2(\omega) + \dots + X_n(\omega))$$

is Poisson$(\lambda_1 + \dots + \lambda_n$).

4.iii) Compute mean and variance of S_n.
Hint: one way is to differentiate the characteristic function at $\theta = 0$.

3.iv) Assuming now that $\lambda_i = 1$ for all $i \in \mathbb{N}$, show that as $n \to \infty$

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \xrightarrow{d} \mathcal{N}(0, 1)$$

(which means convergence in distribution towards a standard gaussian).

Problem 5

On a probability space $(\Omega, \mathcal{F}, P)$, we consider a pair of square integrable random variables $X(\omega), Y(\omega) \in L^2(\Omega, \mathcal{F}, P)$ such that the following **linear regression property** holds for the conditional expectation:

$$E_P(X|\sigma(Y))(\omega) = aY(\omega) + b$$

for some deterministic constants $a, b \in \mathbb{R}$.

5.i) Use the definition of conditional expectation to find the values of a and b in terms of moments and joint moments of X and Y.

5.ii) Assume now that $X_1(\omega), X_2(\omega) \in L^2(\Omega, \mathcal{F}, P)$ are independent and identically distributed random variables.

Let

$$Y(\omega) = X_1(\omega) + X_2(\omega)$$

Show that the linear regression property holds for the conditional expectation $E_P(X_1|\sigma(Y))(\omega)$, find the corresponding values a, b.

Hint: Use the symmetry $E_P(X_1|\sigma(Y))(\omega) = E_P(X_2|\sigma(Y))(\omega)$. which holds since the pairs (X_1, Y) and (X_2, Y) have the same probability distribution.

5.iii) Assume now that X_1 and X_2 are independent and identically distributed Gaussian random variables with 0 mean and variance 1, and let $Y = X_1 + X_2$.

Use Bayes' formula to compute the conditional density of the regular conditional distribution

$$P(X_1 \in dx|\sigma(Y))(\omega)$$

Hint: we recall that a standard gaussian r.v. X with 0 mean and variance 1 has distribution

$$P(X \in dt) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{t^2}{2}\Big) dt$$