Probability theory 7.3.2006

In every exam, the random variables are supposed to be defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

1. Let

$$\mathcal{C} = \sigma((-\infty, x] \times (-\infty, y]; x, y \in \mathbf{R})$$

be a σ-algebra on $\mathbf{R}^2$. Prove that $A \times \mathbf{R} \in \mathcal{C}$ for every Borel set $A \subseteq \mathbf{R}$.

2. Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with the parameter $\mu > 0$. That is, $\mathbf{P}(X_n \leq x) = 1 - e^{-\mu x}, \forall x \geq 0$. Denote $Y_n = \max(X_1, \ldots, X_n)$.

a) Let $M > 0$ be fixed. Show that $\mathbf{P}(Y_n \leq M \text{ i.o.}) = 0$.

b) Show that $Y_n \to \infty$ a.s.

3. Let $\mathbf{P}(\mid X \mid \leq M) = 1$ for a fixed $M > 0$, and let $c(t) = \log \mathbf{E}(e^{tX}), \forall t \in \mathbf{R}$. Show that c is convex, namely, that

$$c(\alpha t + (1-\alpha)u) \leq \alpha c(t) + (1-\alpha)c(u)$$

for every $t, u \in \mathbf{R}$ and $\alpha \in (0,1)$.

4. Let $Y_1, Y_2 \in L$ and $a_1, a_2 \in \mathbf{R}$. Let further $\mathcal{D}$ be a subsigma-algebra of $\mathcal{A}$. Show that

$$\mathbf{E}(a_1 Y_1 + a_2 Y_2 \mid \mathcal{D}) = a_1 \mathbf{E}(Y_1 \mid \mathcal{D}) + a_2 \mathbf{E}(Y_2 \mid \mathcal{D}) \text{ a.s.}$$

5. Let X_j have the Poisson distribution with the parameter $\lambda_j > 0$, $j = 1, 2, \ldots$. That is,

$$\mathbf{P}(X_j = k) = e^{-\lambda_j} \frac{\lambda_j^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Assume that $X_1, X_2, \ldots$ are independent and that

$$\lim_{n\to\infty} \frac{\lambda_1 + \cdots + \lambda_n}{n} = \lambda \in (0, \infty).$$

Prove that

$$\frac{X_1 + \cdots + X_n - (\lambda_1 + \cdots + \lambda_n)}{\sqrt{n}}$$

converges in distribution and derive the limit distribution.

Probability theory 18.5.2006

In every exam, the random variables are supposed to be defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

1. Let $A_1, A_2, \ldots \in \mathcal{A}$. Show that

$$\limsup_{n\to\infty} \mathbf{P}(A_n) \leq \mathbf{P}(\limsup_{n\to\infty} A_n).$$

2. Prove that if $X_n \to X$ in probability then also $X_n \to X$ in distribution.

3. Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables. Let $S_n = X_1 + \cdots + X_n$ for $n = 1, 2, \ldots$. Suppose that $\mathbf{E}(X^4)$ is finite. Let $\mu = \mathbf{E}(X)$ and let $\delta \in (0,1)$ be fixed. Show that

$$\mathbf{P}\left(\lim_{n\to\infty}\left(\sup\left\{\left|\frac{S_{\lfloor nt\rfloor}}{n} - \mu t\right| ; t \in [\delta, 1]\right\}\right) = 0\right) = 1,$$

where $\lfloor c \rfloor$ denotes the biggest integer $\leq c$.

4. Let X be a random variable such that $X(\omega) \in (0,1)$, $\forall \omega \in \Omega$ and let $\mu = \mathbf{E}(X)$. Suppose that the conditional distribution of Y with respect to X is Bernoulli distribution with the parameter X. That is

$$\mathbf{P}(Y \leq y \mid \sigma(X)) = \begin{cases} 0 \text{ for } y < 0 \\ 1 - X \text{ for } y \in [0,1) \\ 1 \text{ for } y \geq 1. \end{cases}$$

Determine $\mathbf{E}(\exp(tY))$ for every $t \in \mathbf{R}$.

5. Let $X_1, X_2, \ldots$ be independent eksponentially distributed random variables with the parameter $\mu > 0$ (the density is $\mu \exp(-\mu z)$ for $z > 0$). Let $Y_n = X_1 + \cdots + X_{n-1} + 2X_n, n = 1, 2, \ldots$. Determine

$$c(t) =: \lim_{n\to\infty} n^{-1} \log \mathbf{E}(\exp(tY_n))$$

for every $t \in \mathbf{R}$ and calculate the convex conjugate c^*. Do we have

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}(n^{-1} Y_n \in F) \leq -\inf\{c^*(x) ; x \in F\}$$

for every closed $F \subseteq \mathbf{R}$.