Examination in Principal bundles and Yang-Mills theory (fall 2006)

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0. Do 4 of the following 5 problems.

1. Show that the duality operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ for differential forms on a n dimensional Riemann manifold M satisfies $** = \pm 1$ and determine the sign as a function of n, k.

2. Let $D = d + H\wedge$ be the twisted differential acting on differential forms on a manifold M with H a closed form of degree 3. Show that $D^2 = 0$ and thus we can define a twisted cohomology theory $H^*_D(M) = \mathrm{ker}D/\mathrm{im}D$ for odd (even) degree forms. Show that $H^*_D(M) = 0$ when $M = S^3$ and compute also $H^*_D(M)$ for the 3-torus $M = S^1 \times S^1 \times S^1$. Here H is the basic 3-form on M such that its integral over the whole manifold is equal to 1. Hint: You are free to use the known ordinary cohomology groups of spheres and the 3-torus.

3. Determine the Lie algebras for the groups $SL(2, \mathbb{R})$, of real 2×2 matrices with unit determinant, and $SO(2,1)$, the group of real 3×3 matrices with unit determinant preserving a pseudo-metric of signature $++-$. Show that they are isomorphic.

4. Show that principal $SU(n)$ bundles over the 4-dimensional unit sphere are classified by the integral of the Chern class in dimension four, by relating the value of the Chern class to a winding number of the transition function.

5. Construct explicitly the Dirac operator on the unit sphere S^2. Define a magnetic field on S^2 such that when its vector potential is coupled to the Dirac operator then the resulting operator has nonzero Fredholm index.