Operator Algebras Exam 2008–12–16

1. Let X be a C^*-algebra. State the definition of positive elements in X. Denote by X^+ the set of all positive elements in X. Prove the following statements:
   (a) $X^+ = \{x^*x : x \in X\}$;
   (b) if $x, y \in X$ are self-adjoint and $z \in X$, then
   $$x \leq y \Rightarrow z^*xz \leq z^*yz;$$
   (c) if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$;
   (d) if $x, y \in X^+$ are invertible, then
   $$x \leq y \Rightarrow 0 \leq y^{-1} \leq x^{-1}.$$
   (Hint: the Gelfand transform and the spectral mapping theorem.)

2. State the spectral theorem for normal operators. Let T be a normal operator on a Hilbert space H and let E be the spectral measure relative to $(\sigma(T), H)$. Show that T is compact if and only if $E(\Delta_\epsilon)$ has finite rank for all $\epsilon > 0$, where $\Delta_\epsilon = \{z : |z| > \epsilon\}$. (Hint: for sufficiency, consider $T - TE(\Delta_\epsilon)$. For necessity, define $f(z) = z^{-1}\chi_{\Delta_\epsilon}(z)$.)

3. Define the Bergman space A^2 for the unit disk $\mathbb{D}$ and the Toeplitz operator T_a acting on A^2 when $a \in L^\infty(\mathbb{D})$. Find the adjoint of T_a.

   Denote by τ the C^*-algebra of bounded linear operators on A^2 generated by $\{T_a : a \in C(\overline{\mathbb{D}})\}$. Show that $\tau = \{T_a + K : a \in C(\overline{\mathbb{D}}),\ K \text{ is compact}\}$. (Hint 1: does τ have any nontrivial reducing subspaces? Hint 2: you could apply the following two results. For $a \in C(\overline{\mathbb{D}})$, the Toeplitz operator T_a is compact on A^2 if and only if $a(z) = 0$ whenever $|z| = 1$. The Hankel operator $H_a = QM_a = (I - P)M_a$ is compact whenever $a \in C(\overline{\mathbb{D}})$. Hint 3: define $\varphi : C(\overline{\mathbb{D}}) \to \tau/\mathcal{K}(A^2)$ by
   $$\varphi(a) = T_a + \mathcal{K}(A^2)$$
   for $a \in C(\overline{\mathbb{D}})$. Is φ a $*$-homomorphism?)

4. Let H be a Hilbert space. Show that a bounded linear operator T is positive if and only if $(Tx, x) \geq 0$ for all $x \in H$.

   Let X be a C^*-algebra and $x \in X$. Suppose that $\tau(x) \geq 0$ for all positive linear functionals τ on X. Is the element x then necessarily positive? (Hint: the GNS construction and the preceding observation.)

5. Let H be a separable infinite-dimensional Hilbert space. We say that an ideal of bounded linear operators on H is proper if there is a bounded linear operator on H that is not contained in the ideal. Give two examples of proper ideals of bounded linear operators on H.

   Show that the algebra of compact operators is the only nonzero closed proper ideal of operators on H.

Operator Algebras Exam 2009–03–03

1. Let H be a Hilbert space and let P be a positive bounded linear operator on H. Show that there exists a unique positive bounded linear operator Q on H such that $Q^2 = P$; also show that the operator Q commutes with each operator that commutes with P. (Hint: the functional calculus.)

2. State the spectral theorem for normal operators and sketch the proof of one of the following results.
   (a) If T is a normal operator on a Hilbert space H, then
   $$\|T\| = \sup_{\|x\| \leq 1} |(Tx, x)|.$$
   (Hint: consider $B = \sigma(T) \cap \{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\}$, where $\lambda \in \sigma(T)$, and define $f(\mu) = (\mu - \lambda)\chi_B(\mu)$. If E is the spectral measure for T, is $E(B)$ nonzero?)
   (b) If T is a normal operator on H, then T has a nontrivial closed invariant subspace. (Hint: consider first the case in which $\sigma(T)$ contains only one point.)

3. Let X be a C^*-algebra and $x \in X$ be self-adjoint. Show that $\sigma(x) \subset \mathbb{R}$. Suppose Y is a C^*-subalgebra of X containing the unit of X and let $y \in Y$. What can you say about the dependence of the spectrum of y on the algebra, i.e., do $\sigma_X(y)$ and $\sigma_Y(y)$ coincide?

4. Define the Bergman space A^2 and the Toeplitz operator $T_a : A^2 \to A^2$ with $a \in L^\infty$. Derive the integral representation of T_a.
   Denote by τ the C^*-algebra of bounded linear operators on A^2 generated by the set of Toeplitz operators $\{T_a : a \in C(\overline{\mathbb{D}})\}$. Recall that a closed ideal I of a C^*-algebra X is called the commutator ideal of X if it is generated by the commutators $[x, y] = xy - yx$ $(x, y \in X)$. Show that the commutator ideal of the Toeplitz algebra τ is the algebra of compact operators. (Hint: recall the characterization of τ in terms of Toeplitz operators and compact operators.)

5. Let X be a C^*-algebra with identity 1 and let $u, p, q \in X$. Suppose that u is invertible with $u^{-1} = u^*$, $p^2 = p = p^*$, $q^2 = q = q^*$, and $p + q = 1$. Show that $pup + q$ is left-invertible if and only if $\|qup\| < 1$; and that $pup + q$ is right-invertible if and only if $\|puq\| < 1$. (Recall that $x \in X$ is said to be left-invertible if there is an element $y \in X$ such that $yx = 1$.)