Matematiikan ja tilastotieteen laitos
Mitta ja integraali
Final Exam
18.12.2007

1. Based on the definition of the Lebesgue outer measure show that

$$m^*(\{x, y) \in \mathbb{R}^2 : x > 1, 0 < y < x^{-2}\}) < \infty.$$

2. Define measurable set. Show that if $A \subset \mathbb{R}^n$ and $m^*(A) = 0$, then A is measurable.

3. Define measurable function. Show that if $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are measurable functions, then $h, h(x) = \max\{f(x), g(x)\}$, is measurable.

4. Determine the limit

$$\lim_{i \to \infty} i \int_1^\infty x^{-3} \sin(x/i) \cos(x/i) dx$$

and prove your claim.

5. Let $E_1, E_2, \dots \subset \mathbb{R}^n$ be measurable sets such that $E_i \subset B(0, 1)$ for all $i = 1, 2, \dots,$ and every point in $\mathbb{R}^n$ belongs to at most three different sets E_i. Show that

$$\sum_{i=1}^{\infty} m(E_i) < \infty.$$

Department of Mathematics and Statistics
Measure and Integral
Final exam
June 12, 2008

1. (a) Define the following notions:
   (i) *the n-dimensional Lebesgue outer measure m_n^*.*
   (ii) *a Lebesgue-measurable set $E \subset \mathbb{R}^n$.*

   (b) Let $A \subset \mathbb{R}^n$ be an arbitrary set. Prove that
$$m_n^*(A) = \inf\{m_n(U) \colon A \subset U,\ U \text{ open}\}.$$

2. Let $(E_i)_{i=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^n$ such that
$$\sum_{i=1}^{\infty} m(E_i) < \infty.$$
Prove that the set $\{x \in \mathbb{R}^n \colon x \in E_i \text{ for infinitely many } i \in \mathbb{N}\}$ is of measure zero.

3. (a) Suppose that $A \subset \mathbb{R}^n$ and let $f \colon A \to \mathbb{R}$ and $g \colon A \to \mathbb{R}$ be measurable functions. Show that the sets $\{x \in A \colon f(x) < g(x)\}$ and $\{x \in A \colon f(x) = g(x)\}$ are measurable.

   (b) Suppose that functions $f_k \colon A \to \mathbb{R}$, $k \in \mathbb{N}$, are measurable. Prove that the set
$$\{x \in A \colon \exists \lim_{k\to\infty} f_k(x) \in \mathbb{R}\}$$
is measurable.

4. Let $f \colon E \to \mathbb{R}$ be a measurable function, where E is a measurable set such that $m(E) < \infty$. Let $E_i = \{x \in E \colon 0 < f(x) < 1/i\}$ for $i = 1, 2, \ldots$. Prove that $\lim_{i\to\infty} m(E_i) = 0$.

5. Find the limit
$$\lim_{k\to\infty} k \int_0^1 x^{3/2} \sin \frac{x}{k}\, dx.$$
[If you rely on some convergence theorems, please, justify their use carefully.]