Measure and Integral

Final exam, 21.3.2005

By measurabilility we always mean measurability with respect to the σ–algebra of Lebesgue–measurable sets.

1. (a) Give the definition of a measurable set.

   (b) Show that given any $\varepsilon > 0$ and $E \subset \mathbb{R}^n$:n measurable, then there exists a closed $F \subset E$ such that $m(E \setminus F) < \varepsilon$.

   Hint: Look at the compelment of E.

2. Assume that $E \subset \mathbb{R}^n$ is measurable, and that it can be written as a union $E = E_1 \cup E_2$, where E_1 ja E_2 are measurable. Show that a function $f : E \to \mathbb{R}$ is measurable if and only if the restrictions $f|_{E_1}$ ja $f|_{E_2}$ are measurable.

3. (From the lectures.) Assume that (f_j) is a sequence of measurable mappings. Show that also

$$f = \limsup_{j\to\infty} f_j$$

   is measurable; You may assume known that a function is measurable if and only if the preimages of intervals $(-\infty, \alpha]$ and preimages of points $\pm\infty$ are measurable.

4. (From the exercises.) Determine the limit

$$\lim_{k\to\infty} \int_0^{\pi} \frac{1}{\sqrt{x + \sin^k x}}\, dx.$$

5. Determine the limit

$$\lim_{k\to\infty} \int_{-\pi/2}^{\pi/2} \left\{ \left(\frac{\sin x}{x} \right)^k + x \right\} dx.$$

Measure and integral

Final exam 25.10.2005

Measurable means always measurability with respect to the σ–algebra of Lebesgue–measurable sets of $\mathbb{R}^n$.

1. (a) What is the definition of a measurable set?

   (b) Show that if the outer measure of A is zero, then it is measurable.

2. Define $f : [0,1] \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x) = n, \quad \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n = 1, 2, \ldots.$$

   Is f measurable? Explain.

3. Compute the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{e^{-n\sin^2 x}}{1+x^2}\, dx.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let

$$A_n = \{x;\ 2^{n-1} < |f(x)| \leq 2^n\}.$$

   Show that f is integrable if and only if

$$\sum_{n=-\infty}^{\infty} 2^n m(A_n) < \infty.$$

5. What is inequality in Fatou's lemma? Give an example where there is a strict inequality in Fatou's lemma.

Matematiikan ja tilastotieteen laitos
Mitta ja integraali
Final exam
7.3.2006

1.Define Lebesgue outer measure in $\mathbb{R}^n$. Show that if $A \subset \mathbb{R}^n$ and $\epsilon > 0$, then there is an open set G such that $A \subset G$ and

$$m^*(G) \leq m^*(A) + \epsilon.$$

2. Define measurable set and measurable function. Show that a set $E \subset \mathbb{R}^n$ is measurable if and only if its characteristic function χ_E is measurable.

3. Let $f : E \to \mathbb{R}$ be a measurable function where E is a measurable set for which $m(E) < \infty$, and let $E_i = \{x \in E : 0 < f(x) < 1/i\}$ for $i = 1, 2, \ldots$. Show that $\lim_{i \to \infty} m(E_i) = 0$.

4. Determine

$$\lim_{i \to \infty} \int_0^2 x \cos(1/(ix)) e^{-x^i} dx$$

and prove your claim.

5. Let $E \subset \mathbb{R}^n$ be a measurable set for which $m(E) < \infty$, and let $f : E \to \mathbb{R}$ be a measurable function such that f^2 is integrable over E. Show that also f integrable over E.

Matematiikan ja tilastotieteen laitos
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Final exam
6.3.2007

1. Define Lebesgue outer measure. Is it true for $A \subset \mathbb{R}^n$ that
a) if $m^*(A) > 0$, then A contains a non-empty open interval,
b) if A is bounded, then $m^*(A) < \infty$?
Justify your claims.

2. Define measurable set. Prove that if $A \subset \mathbb{R}^n$ and if for every $\epsilon > 0$ there is a measurable set $B \subset A$ such that $m^*(A \setminus B) < \epsilon$, then A is measurable.

3. Let $A \subset \mathbb{R}^2$ be a measurable set. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = m_2(\{(x, y) \in A : t < x < t + 1\}), t \in \mathbb{R}.$$

Show that f is a measurable function.

4. Determine the limit

$$\lim_{i \to \infty} \int_0^1 x^{-\frac{1}{2}} e^{x/i} \cos(x/i) dx,$$

and justify your claim.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Show that $\liminf_{i \to \infty} i \int_i^{i+1} |f| = 0$. Is it also always true that $\liminf_{i \to \infty} i \int_i^{i+1} f = 0$?