Mathematics of Spatial Population Dynamics
Exam 20.4.2004.

Answer to all four questions. You may give your answers in Finnish or in English.

1. Assume that $v = v(x, t; y)$ satisfies the equation

$$v_t(x, t; y) = v_{xx}(x, t; y) - v(x, t; y) \tag{1}$$

in the domain $x \in \Omega = (-1, 1)$ with the initial condition $v(x, 0; y) = \delta_y(x)$ (the delta distribution concentrated at y, where y is a point in Ω) and the reflecting boundary conditions $v_x(\pm 1, t; y) = 0$. Let $T(y)$ be defined as

$$T(y) = \int_X \int_0^\infty v(x, t; y) dt dx, \tag{2}$$

where $X \subset \Omega$ is the subdomain $X = (0, 1)$. Use integration by parts to derive a differential equation from which $T(y)$ can be solved, and find the value of $T(0)$. What biological quantities could v and T represent?

2. Consider the metapopulation model

$$\frac{dp}{dt} = C(p)(1 - p) - E(p)p, \tag{3}$$

where $p = p(t)$ is the fraction of occupied patches, and the colonization and extinction processes are given by $C(p) = cp(1 + p)$, $E(p) = e/(1 + p)$. What are the invasion and persistence capacities of the model, and to what kind of threshold behavior do they relate to?

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 1 & 3 & 0 \\ 1 & 4 & 1 & 6 \end{pmatrix} \tag{4}$$

What can you say about the eigenvalues and eigenvectors of the matrix A?

4. Consider the following model that describes the dynamics of a plant population in an infinitely large homogeneous 2-dimensional landscape (here $x, y \in \mathbf{R}^2$).

   Let $P(x)$ denote the density of plants at a location x. Each plant individual is assumed to produce offspring into its surroundings at a rate c. The offspring are distributed according to a dispersal kernel $D(x)$, which is radially symmetric and normalized as $\int_{\mathbf{R}^2} D(x)dx = 1$. The death rate of a plant individual at a location x is $\mu + \alpha \int P(y)K(x - y)dy$, where μ is the density-independent death rate, and α is a parameter that relates to density-dependent death. $K(x)$ is a competition kernel, which is radially symmetric and normalized as $\int_{\mathbf{R}^2} K(x)dx = 1$.

   Write down spatial moment equations that describe the time evolution for the density of plants (N_P) and pairs of plants ($C_{PP}(y)$). Derive a mean-field model by ignoring spatial correlations. What is the equilibrium density of plants in the mean-field model?

Mathematics of Spatial Population Dynamics
Exam 18.6.2004.

Answer to all four questions. You may give your answers in Finnish or in English.

1. Assume that $v = v(x, t; y)$ satisfies the equation

$$v_t(x, t; y) = v_{xx}(x, t; y) - v(x, t; y) \tag{1}$$

in the domain $x \in \Omega = (-\infty, \infty)$ with the initial condition $v(x, 0; y) = \delta_y(x)$ (the delta distribution concentrated at y, where y is a point in Ω) and the boundary conditions $v(\pm\infty, t; y) = 0$. Let $T(y)$ be defined as

$$T(y) = \int_X \int_0^\infty v(x, t; y) dt dx, \tag{2}$$

where $X \subset \Omega$ is the subdomain $X = (-1, 1)$. Use integration by parts to derive a differential equation from which $T(y)$ can be solved, and find the value of $T(0)$. What biological quantities could v and T represent?

2. Consider the metapopulation model

$$\frac{dp}{dt} = C(p)(1 - p) - E(p)p, \tag{3}$$

where $p = p(t)$ is the fraction of occupied patches, and the colonization and extinction processes are given by $C(p) = cp^2$, $E(p) = e/(1 + p)$. What are the invasion and persistence capacities of the model, and to what kind of threshold behavior do they relate to? To which model class does the model belong to: Levins type model, a model with a weak Allee-effect, or a model with a strong Allee-effect?

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 1 & 3 & 0 \\ 1 & 4 & 1 & 6 \end{pmatrix} \tag{4}$$

What can you say about the eigenvalues and eigenvectors of the matrix A?

4. Consider the following model that describes the dynamics of a plant population in an infinitely large homogeneous 2-dimensional landscape (here $x, y \in \mathbf{R}^2$).

Let $P(x)$ denote the density of plants at a location x. Each plant individual is assumed to produce offspring into its surroundings at a rate c. The offspring are distributed according to a dispersal kernel $D(x)$, which is radially symmetric and normalized as $\int_{\mathbf{R}^2} D(x)dx = 1$. The death rate of a plant individual at a location x is $\mu + \alpha \int P(y)K(x - y)dy$, where μ is the density-independent death rate, and α is a parameter that relates to density-dependent death. $K(x)$ is a competition kernel, which is radially symmetric and normalized as $\int_{\mathbf{R}^2} K(x)dx = 1$.

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