Mathematical Modelling – Exam 19/12/2007

Question 1

Describe the following systems by a reaction network of unimolecular and bimolecular reactions. Indicate clearly what symbols you use for what *i*-states. Also give the corresponding system of ordinary differential equations for the population densities.

(a) A predator-prey system in which the predator, after it has found a prey, first has to chase the prey. During this chase there is a constant probability per unit of time that the predator actually captures the prey, but there is also a constant probability per unit of time that the prey escapes. If the prey is captured, then the predator needs an exponentially distributed amount of time to eat and digest its prey before it resumes hunting again.

(b) A system in which each encounter between two adult individuals can lead to the formation of a breeding pair. Such a pair then takes an exponentially amount of time to produce and eventually release a single clutch of n offspring. After the offspring has been released, the breeding pair breaks up into two independent individuals again. The offspring take an exponentially amount of time to become adults.

Question 2

Consider the system

$$\frac{dS}{dt} = \alpha S - \beta S I + \gamma I - \delta S$$

$$\frac{dI}{dt} = \beta S I - \gamma I - (\delta + \varepsilon) I$$

where α, β, γ, δ and ε are positive constants and $S \geq 0$ and $I \geq 0$.

(a) What kind of processes may be described by these equations? Give the corresponding reaction network.

(b) Perform a phase plane analysis of this system and determine the stability of all equilibria. Use local stability analysis if the phase plane analysis does not give enough information about stability.

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Question 3

Consider a population in which every encounter between two individuals may result in a wild chase during which the two individuals move around together as a pair, but in a random, undirected way for an average of T time units before breaking up settling down again. This can be modelled as

$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

$$\frac{\partial B}{\partial t} = +\frac{k}{2}A^2 - \frac{1}{T}B + D\frac{\partial^2 B}{\partial x^2}$$

where A denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another. Using a time-scale argument, this system can be reduced to a single equation for the total population density

$C=A+2B$

with a density-dependent diffusion coefficient. To this end, let $k=O(1/\varepsilon)$ and $1/T=O(1/\varepsilon)$ and $D=O(1)$ for some small dimensionless scaling parameter $\varepsilon\downarrow 0$.

(a) Calculate the positive quasi-equilibrium of the above system as a function of C for $\varepsilon\downarrow 0$, and give the equation for the dynamics of C at this quasi-equilibrium.

(b) Rewrite the equation for C in the form

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(\mathbf{D}(C)\frac{\partial C}{\partial x}\right)$$

where $\mathbf{D}(C)$ can be interpreted as a density-dependent diffusion coefficient. Give an explicit expression for $\mathbf{D}(C)$ and sketch its graph as a function of C.

Question 4

Consider the same system as in Question 3, i.e.,

$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

$$\frac{\partial B}{\partial t} = +\frac{k}{2}A^2 - \frac{1}{T}B + D\frac{\partial^2 B}{\partial x^2}$$

where A again denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another.

(a) Suppose that individuals that are not involved in a chase have a tendency to move out of the way of individuals chasing one another. Modify the above system to take this additional process into account.

(b) Suppose the *modified* system in **(a)** has reflecting boundaries for state A at $x=0$ and $x=L>0$ and absorbing boundaries for state B at $x=0$ and $x=L$. Give the appropriate boundary conditions.

Mathematical Modelling – Exam 04/02/2008

Question 1

The life cycle of many butterfly species is linked to that of a so-called parasitoid, which is another insect species (usually a wasp) that lies its eggs inside the larvae of the butterfly so that the larvae do not develop into a new butterfly but into one or more individuals of the parasitoid instead.

Suppose that *(i)* each adult butterfly produces new larvae at a constant rate. ***(ii)*** If a larva is found by a parasitoid, then the parasitoid deposits a single egg inside the larva, independently of the number of eggs that may already be inside. ***(iii)*** A larva containing zero eggs of the parasitoid develops into an adult butterfly after an exponentially distributed amount of time. ***(iv)*** A larva containing a single egg develops into an adult parasitoid after an exponentially distributed amount of time. ***(v)*** Larvae with two or more eggs inside do not develop into anything but simply die with a constant probability per unit of time, irrespectively of the exact number (≥2) of eggs present. ***(vi)*** The death rate for larvae with only one egg inside is the same as that for larvae without eggs inside. ***(vii)*** Butterflies and parasitoids each have their own species-specific death rate.

(a) What is the smallest collection of *i*-states needed to describe the above system?

(b) Model the above processes as unimolecular or bimolecular reactions.

(c) Give the corresponding population equations.

Question 2

Consider the system

$$\frac{dX}{dt} = \alpha - \varepsilon X - \beta X Y$$

$$\frac{dY}{dt} = \gamma \beta X Y - \delta Y$$

for α, β, γ, δ, $\varepsilon > 0$ and $X \geq 0$ and $Y \geq 0$.

(a) What kind of processes may be described by these equations?

(b) Perform a phase plane analysis of this system and determine the stability of all equilibria. If the phase plane analysis does not give enough information to determine the local stability, then use local stability analysis.

(c) Show that limit cycles are not possible. *Hint:* use the Dulac function $U(X,Y)=1/Y$.

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Question 3

Consider the resource-consumer dynamics

$$\frac{\partial R}{\partial t} = \alpha - \beta R C$$
$$\frac{\partial C}{\partial t} = \gamma \beta R C - \delta C - \frac{\partial}{\partial x}\left(\mu C \frac{\partial R}{\partial x} - \nu C \frac{\partial C}{\partial x} \right)$$

for α, β, γ, δ, μ, ν >0 and for $x \in \mathbb{R}$.

(a) What kind of processes may be described by these equations? In particular, interpret the terms with the partial derivatives with respect to x.

(b) Suppose the dynamics of R is fast compared to that of C. Give the quasi-equilibrium of R as a function of C and determine whether it is stable. Give the partial differential equation for C assuming that R has reached its quasi-equilibrium.

(c) Rewrite the equation for C in the form

$$\frac{\partial C}{\partial t} = f(C) + \frac{\partial}{\partial x}\left(\mathbf{D}(C) \frac{\partial C}{\partial x} \right)$$

for some function $f(C)$, and where $\mathbf{D}(C)$ can be interpreted as a density-dependent diffusion coefficient. Give explicit expressions for $f(C)$ and for $\mathbf{D}(C)$. Sketch the graph of $\mathbf{D}(C)$.

Question 4

Suggest a set of partial differential equations plus boundary conditions for the following system in a one-dimensional spatial domain with reflecting boundaries at x=0 and x=L.

Bacterium (B) cells produce a chemical (C) that spontaneously degrades after some exponentially distributed amount of time. The bacteria move randomly. The chemical diffuses. There are also amoeba (A) cells present, which show positive taxis towards higher concentrations of the chemical, but their movement also contains a random, undirected component. Ignore reproduction and death for the bacteria as well as for the amoebae.