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Mathematical Modelling – Exam ~~26/01~~/2006

Question 1

Describe the following systems by a reaction network of unimolecular and bimolecular reactions. Indicate clearly what symbols you use for what *i*-states. Also give the corresponding system of ordinary differential equations for the population densities.

(a) A predator-prey system in which the predator, after it has found a prey, first has to chase the prey. During this chase there is a constant probability per unit of time that the predator actually captures the prey, but there is also a constant probability per unit of time that the prey escapes. If the prey is captured, then the predator needs an exponentially distributed amount of time to eat and digest its prey before it resumes hunting again.

(b) A system in which each encounter between two adult individuals can lead to the formation of a breeding pair. Such a pair then takes an exponentially amount of time to produce and eventually release a single clutch of n offspring. After the offspring has been released, the breeding pair breaks up into two independent individuals again. The offspring take an exponentially amount of time to become adults.

Question 2

Consider the system

$$\frac{dS}{dt} = \alpha S - \beta S I + \gamma I - \delta S$$

$$\frac{dI}{dt} = \beta S I - \gamma I - (\delta + \varepsilon) I$$

where α, β, γ, δ and ε are positive constants and $S \geq 0$ and $I \geq 0$.

(a) What kind of processes may be described by these equations? Give the corresponding reaction network.

(b) Perform a phase plane analysis of this system and determine the stability of all equilibria. Use local stability analysis if the phase plane analysis does not give enough information about stability.

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Question 3

Consider a population in which every encounter between two individuals may result in a wild chase during which the two individuals move around together as a pair, but in a random, undirected way for an average of T time units before breaking up settling down again. This can be modelled as

$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

$$\frac{\partial B}{\partial t} = +\frac{k}{2}A^2 - \frac{1}{T}B + D\frac{\partial^2 B}{\partial x^2}$$

where A denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another. Using a time-scale argument, this system can be reduced to a single equation for the total population density

$C \equiv A+2B$ $\qquad C = A + 2B$

with a density-dependent diffusion coefficient. To this end, let k=O($1/\varepsilon$) and $1/T$=O($1/\varepsilon$) and D=O(1) for some small dimensionless scaling parameter $\varepsilon \downarrow 0$.

(a) Calculate the positive quasi-equilibrium of the above system as a function of C for $\varepsilon \downarrow 0$, and give the equation for the dynamics of C at this quasi-equilibrium.

(b) Rewrite the equation for C in the form

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(\mathbf{D}(C)\frac{\partial C}{\partial x}\right)$$

where $\mathbf{D}(C)$ can be interpreted as a density-dependent diffusion coefficient. Give an explicit expression for $\mathbf{D}(C)$ and sketch its graph as a function of C.

Question 4

Consider the same system as in Question 3, that is,

$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

$$\frac{\partial B}{\partial t} = +\frac{k}{2}A^2 - \frac{1}{T}B + D\frac{\partial^2 B}{\partial x^2}$$

where A again denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another.

(a) Suppose that individuals that are not involved in a chase have a tendency to move out of the way of individuals chasing one another. Modify the above system to take this additional process into account.

(b) Suppose the modified system in **(a)** has for both types of individuals an absorbing boundary at x=0 and a reflecting boundary at x=L>0. Give the appropriate boundary conditions.

Mathematical Modelling – Exam 18/05/2006

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Describe the following systems by a *reaction network* of uni-molecular and bi-molecular reactions. Indicate clearly what symbols you use for what *i*-states. Also give the *corresponding system of ordinary differential equations* for the population densities.

(a) A predator-prey system in which the predator, after it has found a prey, first has to chase the prey. During this chase there is a constant probability per unit of time that the predator actually captures the prey, but there is also a constant probability per unit of time that the prey escapes. If the prey is captured, then the predator needs an exponentially distributed amount of time to eat and digest its prey before it resumes hunting again.

(b) A system in which each encounter between two adult individuals can lead to the formation of a breeding pair. Such a pair then takes an exponentially amount of time to produce and eventually release a single clutch of n offspring. After the offspring has been released, the breeding pair breaks up into two independent individuals again. The offspring take an exponentially amount of time to become adults.

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Consider the system

$$\frac{dS}{dt} = \alpha S - \beta S I + \gamma I - \delta S$$

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where α, β, γ, δ and ε are positive constants and $S \geq 0$ and $I \geq 0$.

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Question 3

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$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

$$\frac{\partial B}{\partial t} = +\frac{k}{2}A^2 - \frac{1}{T}B + D\frac{\partial^2 B}{\partial x^2}$$

where A denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another. Using a time-scale argument, this system can be reduced to a single equation for the total population density

$$C = A + 2B$$

with a density-dependent diffusion coefficient. To this end, let k=O(1/ε) and $1/T$=O(1/ε) and D=O(1) for some small dimensionless scaling parameter $\varepsilon \downarrow 0$.

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where $\mathbf{D}(C)$ can be interpreted as a density-dependent diffusion coefficient. Give an explicit expression for $\mathbf{D}(C)$ and sketch its graph as a function of C.

Question 4

Consider the same system as in Question 3, i.e.,

$$\frac{\partial A}{\partial t} = -k\,A^2 + \frac{2}{T}B$$

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where A again denotes individuals that are not involved in a chase, and where B denotes pairs of individuals chasing one another.

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