Mathematical Finance, Exam (18.12.07) UH, Department of Mathematics and Statistics

Note 1: you have 4 hours time to do the exam. It is allowed to use the course material and textbooks.

Note 2: If you have passed during the current semester the middle term exam with grade ≥ 3, you are not asked to do Problems 1 and 2. Of course if you like to try them you are welcome to do so, correct answers will rise your grade.

Problem 1. Consider a 1-period market with two stocks, $S_t^{(1)}$ and $S_t^{(2)}$, $t \in \{0,1\}$, but we don't have a riskless bank account available.

At time $t = 1$ the values of the stocks are given by

$$S_1^{(i)} = S_0^{(i)}(1 + R^{(i)}(\omega)), \quad i = 1, 2$$

where under P the random variables are $R^{(i)}$ are independent and identically distributed with values in the set $\{u, d\}$ where $u > d > -1$, and

$$p = P(R^{(i)} = u) = 1 - P(R^{(i)} = d), \quad 0 < p < 1 .$$

Question a): is the market arbitrage free ?

Hint: choose one of the assets as numeraire.

Question b): is the market complete ?

Hint: count the possible states of the market and compare it with the number of instruments.

Question c) Compute the set of arbitrage free prices for the swap option

$$(S_1^{(1)} - S_1^{(2)})^+$$

Problem 2 We extend the market described in Problem 1, by introducing a third riskless asset, a bank account B_t with deterministic return $r > -1$ and dynamics

$B_1 = B_0(1 + r)$.

Question a) Under which condition on r the market $(S^{(1)}, S^{(2)}, B)$ is arbitrage free ?

Question b) Show that when the the market $(S^{(1)}, S^{(2)}, B)$ is arbitrage free it is also complete.

Question c) Compute the price and the hedging strategy of the *swap option*

$$(S_1^{(1)} - S_1^{(2)})^+$$

Question d) Is it possible to hedge the swap option $(S_1^{(1)} - S_1^{(2)})^+$ using only the two stocks $(S^{(1)}, S^{(2)})$, without using the bank account B ?

Problem 3 In discrete time, consider the Cox-Rubinstein-Ross binomial model, with one stock (S_t) and one bond (B_t), where $S_t = S_{t-1}(1 + R_t)$ and R_t are i.i.d. random variables with values in $\{d, u\}$ with $-1 < d < u$ and

$$p = P(R_t = u) = 1 - P(R_t = d), \quad 0 < p < 1 \ .$$

and $B_t = B_{t-1}(1 + r)$, with $d < r < u$.

For simplicity assume $S_0 = B_0 = 1$, and $d = -0.1$, $r = 0.2$, $u = 0.4$, $p = 0.3$.

a) Show that the model arbitrage free and complete.

b) Compute the prices and the hedging strategy for the **european** stop loss option

$$f(S_T) = \max(S_T, 1), \text{ for } T = 2$$

c) Compute the price and the hedging strategy of the corresponding **american** option

$$\Big\{\max(S_t, 1) : t = 0, 1, 2\Big\},$$

Which is the optimal exercise price for the buyer of this american option ?

What is the hedging strategy for the seller of this american option ?

Hint: you can look first at these questions for an american call option.

Problem 4 Consider the Black and Scholes model: in the time interval $[0, T]$, we have a stock price $(S_t : t \in [0, T]) \subset \mathbb{R}_+$ and a bank account $(B_t : t \in [0, T]) \subset \mathbb{R}_+$ which satisfy respectively the SDE

$$S_t = s_0 + \int_0^t S_u \, \mu \, du + \int_0^t S_u \, \sigma \, dW_u \ , \quad s_0 > 0 \ ,$$

and the ODE

$$B_t = 1 + \int_0^t B_u r \, du$$

where $(W_t : t \in [0,T]) \subset \mathbb{R}$ is a standard Brownian motion.

a) Write the SDE satisfied by the discounted stock price $\bar{S}_t = S_t / B_t$ and its explicit solution.

b) Use Girsanov theorem compute the likelihood process $Z_t = dQ_t/dP_t$ where Q is the unique martingale measure equivalent to P for the discounted stock price $\bar{S}_t$.

c) Compute the price and and the hedging strategy of the stop loss option

$$F(\omega) = \max(S_T(\omega), K) \, .$$

d) Compute the price and and the hedging strategy of the option

$$G(\omega) = S_T(\omega)^2 \, .$$

Mathematical Finance, Exam (03.04.08) UH, Department of Mathematics and Statistics

Note 1: you have 4 hours time to do the exam.

We agree that for this time it will be enough to solve correctly 3 problems out of 4 to pass the exam with full score 5/5.

It is allowed to use the course material and textbooks, and also pocket calculators

Problem 1. Consider a 1-period market with two stocks, $S_t^{(1)}$ and $S_t^{(2)}$, $t \in \{0,1\}$, and a bank account $S_t^{(0)}$, on a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ equipped with a probability measure P such that $P(\{\omega_i\}) > 0$ for $i = 1,2,3$.

We assume that $S_1^{(0)} = S_0^{(0)}(1+r)$ with $r = 0.1$, and

$$\begin{aligned} S_1^{(1)}(\omega_1) &= S_0^{(1)}(1+u^{(1)}) \\ S_1^{(1)}(\omega) &= S_0^{(1)}(1+d^{(1)}) \quad \text{if} \quad \omega \in \{\omega_2, \omega_3\} \quad , \\ S_1^{(2)}(\omega) &= S_0^{(2)}(1+d^{(2)}) \quad \text{if} \quad \omega \in \{\omega_1, \omega_2\} \quad , \\ S_1^{(1)}(\omega_3) &= S_0^{(2)}(1+u^{(2)}) \quad . \end{aligned}$$

Let $u^{(1)} = 0.4$, $d^{(1)} = -0.3$, $u^{(2)} = 0.7$, $d^{(2)} = -0.1$, $S_0^{(1)} = S_0^{(2)} = 1$.

i) Prove that the market is complete and arbitrage free.

ii) Compute the price and the hedging strategy of the option

$$F(\omega) = \left\{ S_1^{(1)}(\omega) - S_1^{(2)}(\omega) \right\}^2$$

Problem 2. Consider now a market with two stocks $S_t^{(1)}$ and $S_t^{(2)}$ as in Problem 1 but without the possibilty of using the bank account for trading.

i) Show that this market is arbitrage free.

ii) Is the market complete ?

iii) Compute the set $\mathcal{C}$ of arbitrage free prices for the contingent claim $G = \left\{S_1^{(1)}\right\}^2$.

iv) Choose an arbitrage free price for G, $c \in \mathcal{C}$.

Extend the market by introducing a 3rd asset $S_t^{(3)}$, with $S_0^{(3)} = c$ and $S_1^{(3)}(\omega) = G(\omega) = (S_1^{(1)}(\omega))^2$.

v) Show that the extended market arbitrage free and complete compute, the price and hedging strategy of the contingent claim

$$F(\omega) = \{S_1^{(1)} - S_1^{(2)}\}^2$$

Problem 3 In discrete time, consider the classical Cox-Rubinstein-Ross binomial model, with one stock (S_t) and one bond (B_t), where $S_t = S_{t-1}(1 + R_t)$ and R_t are i.i.d. random variables with values in $\{d, u\}$ with $-1 < d < u$ and

$$p = P(R_t = u) = 1 - P(R_t = d), \quad 0 < p < 1 \; .$$

and $B_t = B_{t-1}(1 + r)$, with $d < r < u$.

For simplicity assume $S_0 = B_0 = 1$, and $d = -0.1$, $r = 0.2$, $u = 0.4$, $p = 0.8$.

i) Compute the unique equivalent martingale measure $\tilde{P}$ for the discounted stock price process.

Consider the american option $\{Y_t : t = 0, 1, 2\}$ with $Y_t = E_P(S_T|S_t)$. (NOTE that this P is not necessaly the martingale measure).

ii) Compute the price, the exercise strategy of the buyer and the exercise strategy for the seller for the american option $\{Y_t : t = 0, 1, 2\}$.

Problem 4 We consider a Black and Scholes model for the currency rate dollar/euro, as the solution of the one dimensional stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t),$$
$$S_0 = 0.63$$

where constant $\mu = 1, \sigma = 0.5$, and (W_t) is a Brownian motion with respect to P.

i) Write and solve the stochastic differential equation for the euro/dollar ratio

$$S'_t := \frac{1}{S_t}.$$

Now assume that you have at your disposal for trading one bank account in dollars with interest rate $r = 0.3$, and one bank account account in euro with interest rate $r' = 0.2$,

ii) Choose first the dollar as numeraire, write the stochastic differential equation for the the discounted euro/dollar value, find the equivalent martingale measure $Q' \sim P$, and write the likelihood process $Z_t = dQ'_t/dP_t$.

iii) Choose the euro as numeraire, write the stochastic differential equation for the the discounted dollar/euro value, find the equivalent martingale measure $Q \sim P$, and write the likelihood process $Z_t = dQ_t/dP_t$.

iv) Choose either dollar or euro as a numeraire, to compute the value of the european call option $F = (S'_T - S'_0)^+$, for $T = 1$.

v) Does the price of this option depend on the numeraire ?

Mathematical Finance , Middle Term Exam , (21.10.08),

Huom. You can use a pocket calculator if you want, but it is not allowed to use books or lecture notes. The duration of the exam is 4 hours. The next lectures of this course will be on monday 27.10.

Part I

In a discrete probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $P(\{\omega_1\}) = 0.6$, $P(\{\omega_2\}) = 0.1$, $P(\{\omega_3\}) = 0.3$, the random variables $S^0(\omega), S^1(\omega)$ are stock prices at time $t = 1$ in the one-period market model. Note that we do not have the possibility to use any other instruments, for example you cannot use another bank account with deterministic return.

The prices of S^0 and S^1 at time $t = 0$ are $\pi^0 = \pi^1 = 1$, respectively. At time $t = 1$ the random variables take values

$$\begin{aligned} S^0(\omega_1) = 1.8, \quad S^0(\omega_2) &= S^0(\omega_3) = 0.8, \\ S^1(\omega_1) = 0.6, \quad S^1(\omega_2) &= 1.6, \quad S^1(\omega_3) = 0.8. \end{aligned}$$

1) Using S_0 as a numeraire. find all equivalent risk-neutral measures $Q \sim P$, and answer the questions:

Is the market free of arbitrage ? Is the market complete ?

2) Find also all equivalent risk-neutral measures $Q \sim P$ with respect to the numeraire S_1.

In the previous market model, let $X(\omega) := \mathbf{1}(\omega = \omega_1)$ be the a digital option which at time $t = 1$ takes value 1 if $\omega = \omega_1$, 0 otherwise.

3) Compute the set of arbitrage free prices for the digital option $X(\omega)$.