Martingales and harmonic analysis – exam 13.5.2008 (3 hours)

Suomeksi kääntöpuolella!

Answer to **four** (4) questions of your own choice. All problems are worth the same score.

On every piece of paper which you return, write down **the name of the course, today's date, your full name** and either your **student number** or your **personal identification number** (also called the social security number).

1. Let $\mathscr{G} \subseteq \mathscr{F}$ be σ-algebras of the set Ω and $\mu : \mathscr{F} \to [0, \infty]$ be a measure for which $(\Omega, \mathscr{G}, \mu)$ is σ-finite. Let $f \in L^1_\sigma(\mathscr{F}, \mu)$ and g be a $\mathscr{G}$-measurable function for which $g \cdot f \in L^1_\sigma(\mathscr{F}, \mu)$. Prove that then
$$\mathbb{E}[g \cdot f | \mathscr{G}] = g \cdot \mathbb{E}[f | \mathscr{G}].$$
You may use all the necessary convergence results without having to prove them.

2. Define Doob's maximal function related to a discrete-time martingale $(f_i)_{i \in \mathbb{Z}}$ and formulate (no need to prove) the related Doob's inequality. Let then $(\mathscr{F}_i)_{i \in \mathbb{Z}}$ be a filtration of the space $(\Omega, \mathscr{F}, \mu)$, where all the measure spaces $(\Omega, \mathscr{F}_i, \mu)$ are σ-finite and
$$\sigma\Big(\bigcup_{i \in \mathbb{Z}} \mathscr{F}_i\Big) = \mathscr{F}.$$
Let $p \in (1, \infty)$ and $f \in L^p(\mathscr{F}, \mu)$. Prove (you may use all the necessary density results) that $\mathbb{E}[f | \mathscr{F}_i] \to f$ pointwise a.e. as $i \to \infty$.

3. Formulate Burkholder's inequality concerning the sign transform of a discrete-time martingale $(f_i)_{i=0}^n$. Give also definitions of the concepts 'biconcave' and 'zigzag martingale'. Prove that the existence of an appropriate biconcave function (you don't need to prove the existence) implies Burkholder's inequality. You may use all the required results concerning the fact that it suffices to prove Burkholder's inequality for certain easier martingales than general ones.

4. Let $p, \beta \in (1, \infty)$ be constants and let $w : \mathbb{R} \to \mathbb{R}$ be a concave function which satisfies
$$w(t) = w(-t) \geq 1 - \beta^p |t|^p \quad \text{for all } t \in \mathbb{R} \qquad \text{and} \qquad w(t) = w(1) \cdot |t|^p \quad \text{when } |t| \geq 1.$$
Prove that then $\beta \geq p' = p/(p-1)$.

5. Define the system of (standard) dyadic intervals and the related Haar functions. What kind of a representation in terms of the Haar functions does a function $f \in L^p(\mathbb{R})$ $(1 < p < \infty)$ have? (The result suffices, you don't need to prove it.) Define Petermichl's dyadic shift and prove that it is a bounded operators in the space $L^p(\mathbb{R})$. You may use all the required results concerning martingales without having to prove them.

6. Let $\phi_1, \ldots, \phi_k$ be trigonometric polynomials, $\phi_j(x) = \sum_{m=-M}^{M} a_j(m) e^{i2\pi m x}$, and let in addition $a_k(0) = 0$. Define the conjugate function ψ_k of ϕ_k. Let $f(t) := \prod_{j=1}^{k} \phi_j(x_j + y_j t)$, where $x, y \in \mathbb{R}^k$. Prove that, with an appropriate choice of y (with all $y_j \neq 0$), the function f is also a trigonometric polynomial, and its conjugate function is $g(t) = \prod_{j=1}^{k-1} \phi_j(x_j + y_j t) \cdot \psi_k(x_k + y_k t)$.