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Department of Mathematics and Statistics
Linear algebra and matrices II
Course examination 12.12.2007

1. Show that the projection of $\mathbb{R}^3$ to the x_1x_2-plane, that is, the transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^2$,
$$L(\mathbf{x}) = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
is linear. Moreover, determine $\mathrm{Ker}(L)$ and $\mathrm{Im}(L)$.

2. Find the least squares solutions of the system $A\mathbf{x} = \mathbf{b}$ for
$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$

3. Let $\mathbb{R}^2$ be equipped with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \mathbf{y}$.
   (a) Find out if the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthogonal with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.
   (b) Use the Gram–Schmidt process for the sequence $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ to find an orthonormal basis of $\mathbb{R}^2$ with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.

4. Determine whether the matrix
$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
is othogonally diagonalizable. If so, find a matrix P that orthogonally diagonalizes the matrix A, and determine P^TAP.

Please answer to the course questionnaire

`http://mathstat.helsinki.fi/kurssit/kysely/index.en.html`

right after the examination!

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Department of Mathematics and Statistics
Linear algebra and matrices II
Course examination 10.12.2008

1. Find the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -5 \\ 0 & 0 & 2 \end{bmatrix}.$$

2. Find the least squares solutions of the system

$$\begin{cases} x_1 + 2x_2 = 3 \\ -x_1 + 4x_2 = -1 \\ x_1 + 2x_2 = 5 \end{cases}.$$

Does the system have *solutions*, this is, do there exist real numbers x_1 and x_2 which solve the system?

3. Let $\mathbb{R}^2$ be equipped with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
   (a) Find out if the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ja $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthogonal with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.
   (b) Use the Gram–Schmidt process for the sequence $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$ to find an orthonormal basis of $\mathbb{R}^2$ with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.

4. (a) Let V and W be vector spaces and let $L\colon V \to W$ be a linear transformation. Show that if $L(\mathbf{x}) = \mathbf{b}$ and $L(\mathbf{y}) = \mathbf{b}$ where $\mathbf{x}, \mathbf{y} \in V$ and $\mathbf{b} \in W$, then $\mathbf{x} - \mathbf{y} \in \mathrm{Ker}(L)$.
   (b) Let $A \in \mathbb{R}^{2\times 2}$ and let $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the formula $L_A(\mathbf{x}) = A\mathbf{x}$. If $L_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathrm{Ker}(L_A) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$, is it possible that also $L_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$?