

1. Show that the projection of  $\mathbb{R}^3$  to the  $x_1x_2$ -plane, that is, the transformation  $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,

$$L(\mathbf{x}) = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

is linear. Moreover, determine  $\text{Ker}(L)$  and  $\text{Im}(L)$ .

2. Find the least squares solutions of the system  $A\mathbf{x} = \mathbf{b}$  for

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$

3. Let  $\mathbb{R}^2$  be equipped with an inner product  $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \mathbf{y}$ .

- (a) Find out if the vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are orthogonal with respect to the inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$ .
- (b) Use the Gram–Schmidt process for the sequence  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$  to find an orthonormal basis of  $\mathbb{R}^2$  with respect to the inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$ .

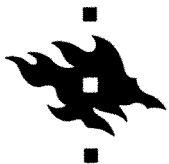
4. Determine whether the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is orthogonally diagonalizable. If so, find a matrix  $P$  that orthogonally diagonalizes the matrix  $A$ , and determine  $P^TAP$ .

Please answer to the course questionnaire

<http://mathstat.helsinki.fi/kurssit/kysely/index.en.html>  
right after the examination!



1. Find the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -5 \\ 0 & 0 & 2 \end{bmatrix}.$$

2. Find the least squares solutions of the system

$$\begin{cases} x_1 + 2x_2 = 3 \\ -x_1 + 4x_2 = -1 \\ x_1 + 2x_2 = 5 \end{cases}$$

Does the system have *solutions*, this is, do there exist real numbers  $x_1$  and  $x_2$  which solve the system?

3. Let  $\mathbb{R}^2$  be equipped with an inner product  $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$ ,  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

- (a) Find out if the vectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  ja  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  are orthogonal with respect to the inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$ .

- (b) Use the Gram–Schmidt process for the sequence  $\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$  to find an orthonormal basis of  $\mathbb{R}^2$  with respect to the inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$ .

4. (a) Let  $V$  and  $W$  be vector spaces and let  $L: V \rightarrow W$  be a linear transformation. Show that if  $L(\mathbf{x}) = \mathbf{b}$  and  $L(\mathbf{y}) = \mathbf{b}$  where  $\mathbf{x}, \mathbf{y} \in V$  and  $\mathbf{b} \in W$ , then  $\mathbf{x} - \mathbf{y} \in \text{Ker}(L)$ .

- (b) Let  $A \in \mathbb{R}^{2 \times 2}$  and let  $L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation given by the formula  $L_A(\mathbf{x}) = A\mathbf{x}$ . If  $L_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  and  $\text{Ker}(L_A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ , is it possible that also  $L_A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ ?