HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Linear algebra and matrices I
Course examination 19.10.2005

1. Solve the linear system

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = -1 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 = 1 \\ 3x_1 - 3x_4 = -3. \end{cases}$$

by Gauss–Jordan elimination.

2. Determine whether the vector $\mathbf{b} = [4 \ \ -1 \ \ 8]^T \in \mathbb{R}^3$ is a linear combination of the vectors $\mathbf{a}_1 = [1 \ \ 2 \ \ -1]^T \in \mathbb{R}^3$ and $\mathbf{a}_2 = [6 \ \ 4 \ \ 2]^T \in \mathbb{R}^3$.

3. Find bases for the row and column spaces of

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

and the rank of the matrix A.

4. (a) Let $A, B, C \in \mathbb{R}^{n\times n}$. Show that if A is invertible and $AB = AC$, then $B = C$.
(b) When $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$, what do you observe of the matrix products AB and AC? Why does your observation not contradict the result of (a) though $B \neq C$?

Please answer to the course questionnaire

`http://mathstat.helsinki.fi/kurssit/kysely/index.en.html`

right after the examination!

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Department of Mathematics and Statistics
Linear algebra and matrices I
Course examination (extra) 26.10.2005

1. Determine whether the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{bmatrix}$$

is invertible, and if so, find its inverse A^{-1}.

2. Determine whether the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$ are linearly independent.

3. Find basis for the nullspace of

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

and the rank of the matrix A.

4. (a) Let $A, B \in \mathbb{R}^{n \times n}$. Show that if $AB = BA$, then

$$(A + B)^2 = A^2 + 2AB + B^2.$$

(b) Give an example of 2×2 -matrices A and B such that

$$(A + B)^2 \neq A^2 + 2AB + B^2.$$

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HELSINGIN YLIOPISTO
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Department of Mathematics and Statistics
Linear algebra and matrices I
Final examination 16.11.2005

1. Determine whether the matrix

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{bmatrix}, \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{bmatrix}$$

is invertible, and if so, find its inverse.

2. Let $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$, $a \neq 0$.
   (a) Prove that
   $$A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix} \quad \text{if } n \geqslant 1.$$
   (b) Determine whether there exist such numbers $a \neq 0$ that A^9 is symmetric or antisymmetric.

3. Let $A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$. Determine numbers a, b and c such that
   (a) $\operatorname{rank}(A) = 1$,
   (b) $\operatorname{rank}(A) = 2$.

4. Determine of the following matrices $A \in \mathbb{R}^{3\times 3}$ whether the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin or the origin only.

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}, \quad \text{(b) } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}, \quad \text{(c) } A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$$

   If the solution space is a line or a plane, find a parametric equation for it.

5. Consider the bases $S = (p_1, p_2)$ and $T = (q_1, q_2)$ for P_1, where
   $$p_1(x) = 6 + 3x, \quad p_2(x) = 10 + 2x, \quad q_1(x) = 2 \quad \text{and} \quad q_2(x) = 3 + 2x.$$
   (a) Find the transition matrices $M(S \leftarrow T)$ and $M(T \leftarrow S)$.
   (b) Find the coordinate vectors $[f]_S$ ja $[f]_T$, where $f(x) = -4 + x$.

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Department of Mathematics and Statistics
Linear algebra ja matrices I
Final examination 18.5.2006

1. Determine whether the matrix

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \text{(b)}\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

is invertible, and if so, find its inverse.

2. Let $\mathbf{v}_1 = [0\ 3\ 1\ -1]^T$, $\mathbf{v}_2 = [6\ 0\ 5\ 1]^T$ and $\mathbf{v}_3 = [4\ -7\ 1\ 3]^T$.
   (a) Show that the sequence $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is linearly dependent in $\mathbb{R}^4$.
   (b) Express each vector $\mathbf{v}_i$ as a linear combination of the other two vectors.

3. (a) Find parametric equation of the line ℓ passing through the points $P = [6\ -1\ 5]^T \in \mathbb{R}^3$ and $Q = [7\ 2\ -4]^T \in \mathbb{R}^3$.
   (b) Find the point $R \in \mathbb{R}^3$ where ℓ intersects the x_2x_3-plane $\{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 = 0\}$.
   (c) Find parametric equation of the plane T passing through the origin O and the points P and Q.

4. Find bases for the row space, the column space and the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

and the rank of the matrix A.

5. Determine whether
   (a) the set A of all $n \times n$ -matrices such that the sum of the main diagonal entries is zero,
   (b) the set B of all antisymmetric (skew-symmetric) $n \times n$ -matrices

   is a subspace of $\mathbb{R}^{n \times n}$.

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Department of Mathematics and Statistics
Linear algebra and matrices I
Course examination 18.10.2006

1. Determine whether the matrix

$$\text{(a) } A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{bmatrix}, \quad \text{(b) } B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

is invertible, and if so, find its inverse matrix.

2. Let $\mathbf{a}_1 = [1\ \ 2\ \ 3]^T, \mathbf{a}_2 = [-2\ \ 1\ \ 0]^T, \mathbf{a}_3 = [1\ \ 0\ \ 1]^T \in \mathbb{R}^3$. Show that the sequence $S = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is a basis for $\mathbb{R}^3$.

3. Find basis for the nullspace of

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$$

and the rank of the matrix A.

4. Matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

has the property that $A^2 = \mathbf{0}$. Is it possible for a symmetric matrix $B \in \mathbb{R}^{2\times 2}$, where $B \neq \mathbf{0}$, to have this property, i.e., $B^2 = \mathbf{0}$? Prove your answer.

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Department of Mathematics and Statistics
Linear algebra and matrices I
Final examination 16.11.2006

1. Use Gauss-Jordan elimination to solve the linear systems

$$\text{(a)} \begin{cases} x_1 + 3x_2 + x_3 + x_4 = 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 8 \\ 3x_1 + x_2 + 2x_3 - x_4 = -1 \end{cases}, \quad \text{(b)} \begin{cases} x_1 - x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - x_3 = 1 \\ 7x_1 + 3x_2 + 4x_3 = 7 \end{cases}.$$

2. Determine whether the set
   (a) $A = \{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 + x_3 = 1\}$,
   (b) $B = \{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$,
   (c) $C = \{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_3 = x_1 + x_2\}$,
   (d) $D = \{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_3 = x_1 \text{ tai } x_3 = x_2\}$,
   form a subspace of the space $\mathbb{R}^3$.

3. Find bases for the row space, the column space and the nullspace of

$$A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

   and the rank of the matrix A.

4. Let $S = (1, x, x^2)$ and $T = (1, 2x, 4x^2 - 2)$ be two bases of the space P_2.
   (a) Find the transition matrix $M(S \leftarrow T)$.
   (b) Find the transition matrix $M(T \leftarrow S)$.
   (c) Determine $[p(x)]_T$ if $p(x) = a + bx + cx^2 \in P_2$.

5. Let $A, B \in \mathbb{R}^{n \times n}$. Show that if both B and AB are invertible, then also A is invertible.

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Department of Mathematics and Statistics
Linear algebra and matrices I
Final examination 15.5.2007

1. Determine whether the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \ a, b, c \in \mathbb{R},$$

are invertible, and if they are, find the inverse.

2. Let P_n be the space of polynomials $p\colon \mathbb{R} \to \mathbb{R}$ of degree $\leqslant n$, $n \in \mathbb{N}$. Determine whether the polynomials
   (a) $x^2 + 1, x^2 - 1$ and $x^2 + x + 1$ span the space P_2,
   (b) $x^3 - 1, x^2 + 1, x - 1$ and 1 span the space P_3,
   (c) $x^3, x^2 + 1, x^2 - x$ and $x + 1$ span the space P_3.

3. Find bases for the row space, the column space and the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{bmatrix}$$

and the rank of the matrix A.

4. Determine of the following matrices $A \in \mathbb{R}^{3\times 3}$ whether the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin or the origin only.

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}, \quad \text{(b) } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}, \quad \text{(c) } A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}.$$

If the solution space is a line or a plane, find a parametric equation for it.

5. The matrix $S \in \mathbb{R}^{n\times n}$ is said to be a *(real) square root* of the matrix $A \in \mathbb{R}^{n\times n}$ if $S^2 = A$.
   (a) Show that $S = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ is a square root of the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$. Are there any other square roots of A?
   (b) Explain why only square matrices can have a square root.
   (c) Find all (real) square roots of the 2×2 identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.