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Lineaarialgebra II
Midterm examination 1, 21. 10. 2005

1. Which of the following subsets of the real vector space $F(\mathbb{R},\mathbb{R}) = \{ f \mid f{:}\,\mathbb{R} \to \mathbb{R} \}$ are its subspaces?:

$$\begin{aligned} U &= \{ x \in F(\mathbb{R},\mathbb{R}) \mid x(1) > 0 \}, \\ V &= \{ x \in F(\mathbb{R},\mathbb{R}) \mid x(t) = x(-2t), \text{ for } t \in \mathbb{R} \}, \\ W &= \{ x \in F(\mathbb{R},\mathbb{R}) \mid x(t)x(-t) = x(0)^2, \text{ for } t \in \mathbb{R} \}. \end{aligned}$$

2. Denote by $\mathcal{P}$ the real vector space of real polynome functions. Put $\pi_k{:}\,\mathbb{R} \to \mathbb{R}$, $\pi_k(t) = t^k$, for $k \in \mathbb{N}$. Show that the set $\{\pi_1, \pi_2, \pi_3\}$ is free.

3. a) Find an example of a module L which does not have a base, i.e, a free generating set.
b) Find an example of a vector space V which is not finitely generated.

4. Consider the real vector space of real number sequences

$$s = F(\mathbb{N},\mathbb{R}) = \{ (a_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}\, (a_n \in \mathbb{R}) \}.$$

Let S be the set of sequences $(a_n)_{n\in\mathbb{N}} \in s$ with $a_n \geq 0$, for $n \in \mathbb{N}$. Determine if there exists a base E for s such that $E \subset S$.

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Department of Mathematics and Statistics
Linear algebra II
Midterm examination 2, 14.12.2005

1. Consider the real vector space $s = F(\mathbb{N}, \mathbb{R})$ and its subspace

$$c_0 = \{\, x \in s \mid \lim_{n\to\infty} x(n) = 0 \,\}.$$

Show that there exist a linear map $A\colon s \to s$ such that $A \restriction c_0 = \mathrm{id}_{c_0}$ though for some $x \in s$, the pair $(x, A(x))$ is free.

2. Let U and Z be finite-dimensional subspaces of the vector space V. Prove that

$$\dim(U + Z) = \dim(U) + \dim(Z) - \dim(U \cap Z).$$

3. Let $\mathcal{P}$ be the real vector space of real polynomial functions and let $A\colon \mathcal{P} \to \mathcal{P}$,

$$A(x)(t) = t(Dx)(t)$$

where D is the derivative. Show that A is a linear map and determine its spectrum $\sigma(A)$.

4. Find examples which show that the cartesian product of vector spaces may be isomorphic or non-isomorphic to the tensor product of the same spaces. In other words: The exist vector spaces (with a common coeffiecent field) U, V, W and Z such that $U \times V \cong U \otimes V$ and $W \times Z \ncong W \otimes Z$.