Lien algebroiden loppukoe 20.5. 2003
Matematiikan laitos

1. Muodosta kaikki reaaliset Lien algebrat, modulo isomorfismit, kun algebran dimensio on kaksi.

2. Osoita, käyttäen hyväksi yleisiä lauseita juuristojen rakenteesta mutta ilman juuristojen luokittelulausetta, että ei ole olemassa puoliyksinkertaisia Lien algebroita joiden dimensio on 5 tai 7.

3. Valitse kanta yksinkertaisen kompleksisen Lien algebran A_2 Cartanin alialgebrasssa $\mathbf{h}$. Selvitä mitkä painot $\lambda \in \mathbf{h}^*$ voivat olla korkeimpia painoja A_2:n äärellisdimensionaalisissa esityksissä. Ilmoita tulos ehtona luvuille $\lambda(h_i)$, missä $\{h_i\}$ on valitsemasi kanta. Mikä on A_2:n redusoitumattoman 10-dimensionaalisen esityksen korkein paino?

4. Olkoon $\mathbf{g} = A_1$. Määrää affiinin Lien algebran $\hat{\mathbf{g}}$ juuret. Osoita, että kiinteällä algebran $\hat{\mathbf{g}}$ keskuksen k arvolla on olemassa vain rajoitettu määrä (mikä?) algebran $\hat{\mathbf{g}}$ ei-ekvivalentteja integroituvia korkeimman painon eistyksiä.

5. Olkoot K, K^{-1}, E, F kvanttialgebran $U_q(\mathbf{sl}(2))$ standardigeneraattorit,

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Tässä $\pm 1 \neq q \in \mathbb{C}$.

Osoita, että alkio $u \in U_q$ on ryhmäkaltainen (group like), siis $\Delta(u) = u \otimes u$, jos ja vain jos $u = K^n$ jollekin $n \in \mathbb{Z}$. Tässä

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes K + 1 \otimes E, \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

LIE ALGEBRAS

Examination January 26th, 2006

Solve 5 problems out of 6!

1. Find all two dimensional Lie algebras, up to isomorphism.

2. Consider the 3-dimensional complex Lie algebra with basis a, b, h and commutation relations

$$[a, b] = -a, \ [h, a] = [h, b] = 0.$$

Construct a faithful 2-dimensional representation of this Lie algebra and find the weights of the representation.

3. Using the root axioms, find the possible values for the angle between a pair of roots.

4. Let $h_1 = e_{11} - e_{22}$ and $h_2 = e_{22} - e_{33}$ be the basis vectors for the Cartan subalgebra of the simple Lie algebra A_2. Here e_{ij} are the standard Weyl basis vectors. What are the posssible values $\lambda_i = \lambda(h_i)$ when λ is an integral dominant weight?

5. Let V_λ be an irreducible module with highest weight λ for the affine Kac–Moody algebra constructed from the simple Lie algebra $\mathbf{sl}(2)$. Assume the hermiticity relations $(T_a^n)^* = -T_a^n$ for the standard basis of the loop algebra. Derive from this the inequality $\lambda(h) \leq k$. The commutation relations are

$$[T_a^n, T_b^m] = \lambda_{ab}^c T_c^{n+m} + kn\delta_{n,-m}\mathrm{tr}(T_a T_b),$$

where the T_a's are the antihermitean 2×2 matrices defining the Lie algebra $\mathbf{sl}(2)$.

6. Construct an example of a commutative but not cocommutative Hopf algebra.

Examination in Lie algebras

April 3, 2006

Department of Mathematics and Statistics

CHOOSE 5 OF THE 6 PROBLEMS!

1. a) Show that any subalgebra of a nilpotent Lie algebra is nilpotent and the image of a nilpotent Lie algebra in any homomorphism is nilpotent. b) Let $\mathbf{k} \subset \mathbf{g}$ be a proper subalgebra of a nilpotent Lie algebra (i.e., $\mathbf{k} \neq \mathbf{g}$) and let $N(\mathbf{g}, \mathbf{k})$ be the set of elements $x \in \mathbf{g}$ such that $[x, y] \in \mathbf{k}$ for all $y \in \mathbf{k}$. Show that $\mathbf{k} \subset N(\mathbf{g}, \mathbf{k})$ is a proper subalgebra.

2. Prove directly from the definition that the set of diagonal traceless matrices is a Cartan subalgebra of $\mathbf{sl}(n, \mathbb{C})$.

3. Let $\Phi \subset E$ be a system of roots. Let $\alpha, \beta \in \Phi$ be a pair of linearly independent roots. Let $E' \subset E$ be the subspace spanned by the roots α, β and let Φ' be the set of roots contained in E. Show that (E', Φ') is a system of roots.

4. Construct an irreducible six dimensional representation of the simple Lie algebra $\mathbf{sl}(3, \mathbb{C})$ and determine the weights of this representation.

5. Let $\hat{\mathbf{g}}$ be the affine algebra $A_1^{(1)}$ obtained from the loop algebra of $\mathbf{sl}(2, \mathbb{C})$ by adjoining the central element k and the derivation d in the standard way. Then an irreducible integrable representation of $\hat{\mathbf{g}}$ is completely determined by the eigenvalue of k and the value $2j$ of the Cartan element $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{sl}(2, \mathbb{C})$ on the highest weight vector. Derive an inequality relating j to k from the conditions $< \lambda, \alpha_i >= 0, 1, 2, \dots$ where λ is the highest weight and α_i's are the simple roots.

6. Explain the Hopf algebra structure in the algebra of functions on a finite group.