Department of Mathematics and Statistics

Introduction to Discrete Mathematics

26.1.2006

Write your name and your social security or student number on each paper.

1. We denote by $D \Delta E$ the *symmetric difference* $(D \setminus E) \cup (E \setminus D)$ of the sets D and E. Show that the equation $|D \Delta E| = |D| + |E| - 2\,|D \cap E|$ holds when D and E are finite.

2. Let X be a set and $A \subset X$. Show that the formula $\varphi(E) = E \Delta A$ defines a bijection $\mathcal{P}(X) \to \mathcal{P}(X)$.

   [Hint: Show that φ is its own inverse mapping.]

3. We define the Fibonacci sequence $F_0, F_1, F_2, \ldots$ recursively by the conditions $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 1$.
   Prove by induction that the following equation holds for every $n \in \mathbb{N}$:

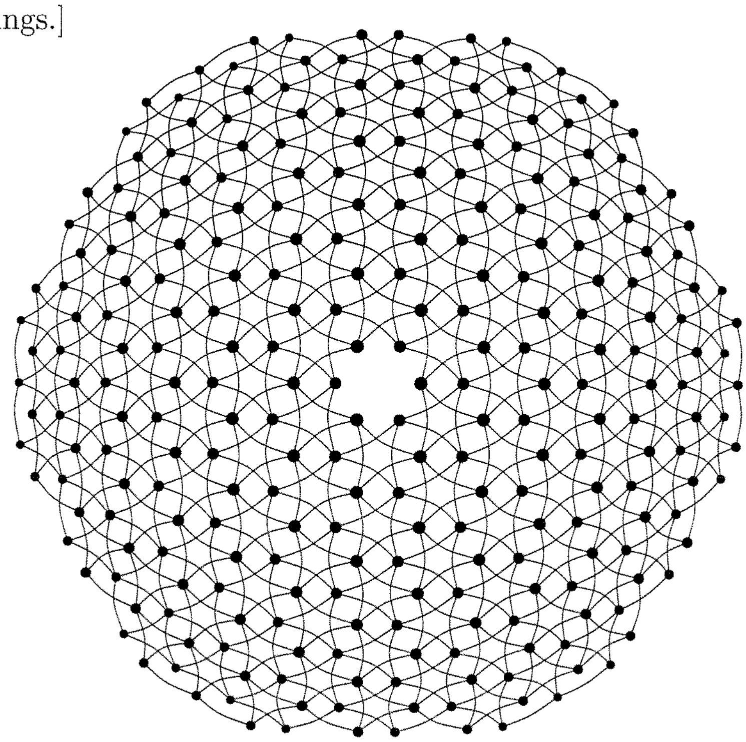
$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1} \,.$$

4. (a) In how many different ways can we place eight distinct balls in seven distinct boxes?
   (b) In how many of the placements in (a) is there at least one ball in every box?

   [Hint: Consider the placements as mappings.]

5. Calculate the number of edges of the graph described on the right. The mere answer is not enough, show also the calculations!

   [Hint: "The graph consists of six similar sectors."]



Department of Mathematics and Statistics

Introduction to Discrete Mathematics

3.4.2006

Write your name and your social security or student number on each paper.

1. We denote by $D\Delta E$ the *symmetric difference* $(D \setminus E) \cup (E \setminus D)$ of the sets D and E. Use Venn diagrams to justify the following equation between sets:

$$(A \cap B)\Delta\left[(B \cap C)\Delta(C \cap A)\right] = (A \cap B) \cup \left[(B \cap C) \cup (C \cap A)\right] .$$

2. We define the mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by the formula

$$f\left((n,k)\right) = \frac{(n+k)(n+k+1)}{2} + n .$$

(a) Show that f is one-to-one.

(b) f is also an onto mapping, but you do *not* need to show this. Instead, study the inverse mapping of f graphically by exhibiting points $\bar{x}_i = (n_i, k_i) \in \mathbb{N} \times \mathbb{N}$ such that $f(\bar{x}_i) = i$ for $i = 0, 1, ..., 9$; based on this consideration, explain how the inverse mapping "behaves".

[*Hint for (a):* Show that $\frac{(h+j)(h+j+1)}{2} + h < \frac{(h+j+1)(h+j+2)}{2}$ and use this to prove that $f(n,k) \neq f(m,\ell)$ whenever $n + k \neq m + \ell$.]

3. Give a combinatorial justification for the following identities (in other words, use the definition of the binomial coefficient $\binom{p}{q}$ instead of the derived formula $\frac{p!}{q!(p-q)!}$).

(a) $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ for every $n = 1, 2, 3, ...$

(b) $\binom{2n}{2} = 2\binom{n}{2} + n^2$ for every $n = 2, 3, 4, ...$

[*Hint:* Split the $2n$-set, whose subsets are being counted, into two n-sets.]

4. Use induction to show that the following equation holds for every $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} k! \cdot k = (n+1)! - 1 .$$

5. By a "binary tree", we mean a tree which has one node of degree 2 and all the other nodes are either leaves or have degree 3.
Show that each binary tree has an odd number of nodes.

Department of Mathematics and Statistics

Introduction to Discrete Mathematics

15.12.2006

Write your name and your social security or student number on each paper.

1. Let f be a mapping $X \to Y$. Show that the following hold for all $A \subset X$ and $B \subset Y$:

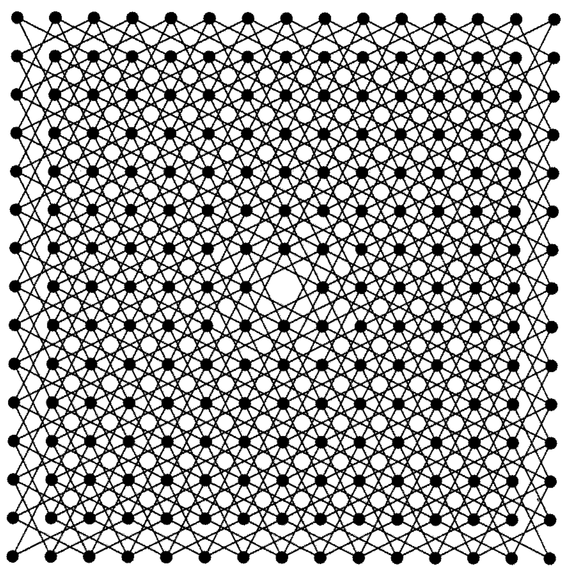
$$A \subset f^{-1}(f(A)) \quad \text{and} \quad f(f^{-1}(B)) \subset B\,.$$

2. We define the Fibonacci sequence $F_0, F_1, F_2, \ldots$ recursively by the conditions $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 1$.
Prove by induction that the following equation holds for every $n = 1, 2, 3, \ldots$:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n\,.$$

3. We form "words" by rearranging the letters of the word MAANALAINEN. In how many of these words, the letters E and I are *not* adjacent?
[For example, MANANALAINE counts, but LAAMANNEINA does not.]
[Hint: From the number of *all* different words, subtract the number of the words containing either EI or IE.]

4. Answer the following questions on the graph G described on the right:
(a) How many edges does G have?
(b_1) Is G bipartite?
(b_2) Is G connected?
(b_3) What is the distance in G between the two "lower corner points"?
Part (a): *Show also the calculations!* *In parts* (b_i), *the mere answer suffices.*



The grading should be completed by the second week of January.

Department of Mathematics and Statistics

Introduction to Discrete Mathematics

April 12, 2007

Write your name and your social security or student number on each paper.

1. (a) Use Venn diagrams to justify the following equation between sets:

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)\,.$$

(b) Let A, B and C be finite. Use the result in (a) to derive the following equation:

$$|A \setminus (B \setminus C)| = |A| - |A \cap B| + |A \cap B \cap C|\,.$$

2. We define the mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by the formula

$$f\left((n,k)\right) = \frac{(n+k)(n+k+1)}{2} + n\,.$$

Show that f is one-to-one.

[*Hint:* Show that $\frac{(h+j)(h+j+1)}{2} + h < \frac{(h+j+1)(h+j+2)}{2}$ and use this to prove that $f(n,k) \neq f(m,\ell)$ whenever $n + k \neq m + \ell$.]

3. We form "words" by rearranging the letters of the word MENNINKÄINEN. How many of these words do *not* contain adjacent letters M and Ä?

[Hint: From the number of *all* different words, subtract the number of the words containing adjacent M and Ä.]

4. We define the Fibonacci sequence $F_0, F_1, F_2, \ldots$ recursively by the conditions $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 1$.
Prove by induction that the following equation holds for every $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1\,.$$

5. Are the two graphs below isomorphic? Justify your answer!