Department of Mathematics and Statistics
Homotopy theory
Final exam 19.12.2006

1. a) Give an example of spaces (with base points) (X, x_0), (Y, y_0) and a function $f\colon (X, x_0) \to (Y, y_0)$ such that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective, but not injective.

b) Give an example of spaces (with base points) (X, x_0), (Y, y_0) and a function $f\colon (X, x_0) \to (Y, y_0)$ such that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective, but not surjective.

c) Give an example of spaces (with base points) (X, x_0), (Y, y_0) and a function $f\colon (X, x_0) \to (Y, y_0)$ such that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism, but X and Y are not homeomorphic.

2. One of the following, a) or b):

a) Prove in detail, that the closed half plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ is not homeomorphic with the plane $\mathbb{R}^2$.

b) What is known about the homotopy groups $\pi_k(S^n)$, for values $1 \leq k \leq n$ (you don't have to prove this). Using this information, prove in detail that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.

3. Suppose that $p\colon (X, x_0) \to (Y, y_0)$ is a covering map, Z a connected and locally path connected space, $z_0 \in Z$ and $f\colon (Z, z_0) \to (Y, y_0)$ continuous. Formulate a condition concerning the fundamental groups, which is a necessary and sufficient condition for the existence of a lifting $\tilde{f}\colon (Z, z_0) \to (X, x_0)$ (you don't have to prove this).

Prove that every continuous mapping $f : \mathbb{R}P^2 \to S^1$ is null homotopic ($\mathbb{R}P^2$ is the projective plane).

4. Suppose X and Y are topological spaces, $f\colon X \to Y$ a homotopy equivalence, $x_0 \in X$ and $y_0 = f(x_0)$. Prove: if y_0 is a good base point (that is: the inclusion $\{y_0\} \to Y$ is a cofibration), then f has a homotopy inverse g for which $g(y_0) = x_0$.

5. Describe briefly the construction of the fundamental group: what are the elements of the group, how are they constructed? How is the group operation defined? Present also the definition and most important properties of the induced homomorphism. Here you don't have to give proofs, if needed you can for example use the phrase "It can be proved that ...".

How can this definition be generalized (the groups π_n)?