University of Helsinki
Department of Mathematics and Statistics
Hierarchical models - spring 2008

Exam 13.5.2008

ANSWER TO 4 QUESTIONS - CREDIT FOR ADDITIONAL ANSWERS

1. Consider a general linear mixed model

$$y_j = \boldsymbol{x}'_{ij}\beta + \boldsymbol{z}'_{ij}\eta_j + \epsilon_{ij},$$

where $\boldsymbol{x}_{ij}$ is a vector of fixed covariates for unit $i, i = 1, ..., n_j$ in cluster $j, j = 1, ..., K$, β is the corresponding vector of regression coefficients, and $\boldsymbol{z}_{ij}$ is the vector of design variables for the latent variables η_j. Assume that $\eta_j \sim N(0, \psi),\ \epsilon_{ij} \sim N(0, \theta)$.

a) Write down the hierarchical representation and reduced form of a *random intercept model* using appropriate choices of the above general model.
b) What are the model assumptions?
c) What is meant by local independence in the model?
d) What is the interpretation of the parameters (both regression parameters β and 2-level parameters)?

2. Consider a latent variable linear model where

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\eta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\eta} = \mu_\eta + \boldsymbol{\zeta},\ \boldsymbol{\epsilon} \sim N(0, \boldsymbol{\Theta}), \boldsymbol{\zeta} \sim N(0, \boldsymbol{\Psi})$$

a) What is the posterior expectation of the latent variables $E(\boldsymbol{\zeta}|\boldsymbol{y})$? How do you interpret the result?

b) What is the posterior expectation of η_j in the random intercept model in Q1?

Hint: Use the results of multinormal distribution for random variables X and Z: Let $(X, Z) \sim N(\mu, \Sigma)$. Then $X|Z \sim N(\mu_{X|Z}, \Sigma_{X|Z})$, where $\mu_{X|Z} = \mu_x + \Sigma_{XZ}\Sigma_{ZZ}^{-1}(Z - \mu_Z)$ and $\Sigma_{X|Z} = \Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$. Note: what are your X and Z here?

3. The estimator of the regression parameter vector of a linear model $\boldsymbol{y}_j = \boldsymbol{X}_j\boldsymbol{\beta}_j + \boldsymbol{\epsilon}_j$ is a linear function of the observations $\boldsymbol{y}_j$, and of the form $\hat{\boldsymbol{\beta}}_j = (\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\boldsymbol{y}_j$ with covariance $cov(\hat{\boldsymbol{\beta}}_j) = \sigma^2(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}$. The estimator is therefore normally distributed and can be written as

$$\hat{\boldsymbol{\beta}}_j = \boldsymbol{\beta}_j + \boldsymbol{r}_j,$$

where the residuals $\boldsymbol{r}_j \sim N(0, \sigma^2(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1})$.

In a hierarchical representation, the 2-level model for the random coeffcients $\boldsymbol{\beta}_j$ is

$$\boldsymbol{\beta}_j = \boldsymbol{\gamma}'\boldsymbol{w}_j + \boldsymbol{\zeta}_j, j = 1, ..., K$$

where $\boldsymbol{w}_j$ are cluster-level covariates and the 2-level residuals are $\boldsymbol{\zeta}_j \sim N(0, \boldsymbol{\Psi})$. Find the variance of the estimator $\hat{\boldsymbol{\beta}}_j$.
Hint: Write the model in reduced form and 'collect' all variation related to the estimation of $\boldsymbol{\beta}_j$.

4. The weighted empirical Bayes estimator of the regression parameter vector in the above linear hierarchical model is

$$\boldsymbol{\beta}_j^{EB} = \omega_j \hat{\boldsymbol{\beta}}_j + (1 - \omega_j)\boldsymbol{w}_j \hat{\gamma}, j = 1, ..., K,$$

where $\hat{\boldsymbol{\beta}}_j$ is the ordinary 'non-hierarchical' least-squares estimator and $\boldsymbol{w}_j \hat{\gamma}$ the hierarchical model estimator of the model in Q3.

Consider again the random intercept model with one covariate.
a) Write down the weighted empirical Bayes estimator for random intercepts?

b) What is the form of the weight ω_j (or $1 - \omega_j$) and its interpretation?

c) How would aggregation (use of cluster means only) affect the EB regression coefficients? How would 'non-hierarchical' modelling (neglecting clustering) affect them?

5. Use of latent variables in hierarchical modelling

6. Classical latent variable models (common structure and differences)

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Exam 20.5.2008

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1. Consider a general linear mixed model

$$y_j = \boldsymbol{x}'_{ij}\beta + \boldsymbol{z}'_{ij}\eta_j + \epsilon_{ij},$$

where $\boldsymbol{x}_{ij}$ is a vector of fixed covariates for unit $i, i = 1, ..., n_j$ in cluster $j, j = 1, ..., K$, β is the corresponding vector of regression coefficients, and $\boldsymbol{z}_{ij}$ is the vector of design variables for the latent variables η_j. Assume that $\eta_j \sim N(\mu_\eta, \psi)$, $\epsilon_{ij} \sim N(0, \theta)$.

a) Write down the hierarchical representation and reduced form of a *random coefficients model* using appropriate choices of the above general model.
b) What are the model assumptions?
c) What is meant by local independence in the model?
d) What is the interpretation of the parameters (both regression parameters β and 2-level parameters)?

2. Consider a latent variable linear model where

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\eta} + \boldsymbol{\epsilon},\ \boldsymbol{\eta} = \mu_\eta + \boldsymbol{\zeta},\ \boldsymbol{\epsilon} \sim N(0, \boldsymbol{\Theta}),\ \boldsymbol{\zeta} \sim N(0, \boldsymbol{\Psi})$$

a) What is the posterior expectation $E(\boldsymbol{\zeta}|\boldsymbol{y})$? How do you interpret the result?

b) What is the posterior expectation of η_j in the random intercept model in Q1?

Hint: Use the results of multinormal distribution: Let $(X, Z) \sim N(\mu, \Sigma)$. Then $X|Z \sim N(\mu_{X|Z}, \Sigma_{X|Z})$, where $\mu_{X|Z} = \mu_x + \Sigma_{XZ}\Sigma_{ZZ}^{-1}(Z - \mu_Z)$ and $\Sigma_{X|Z} = \Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$.

3. The estimator of the regression parameter vector of a linear model $\boldsymbol{y}_j = \boldsymbol{X}_j\boldsymbol{\beta}_j + \boldsymbol{\epsilon}_j$ is a linear function of the observations $\boldsymbol{y}_j$, and of the form $\hat{\boldsymbol{\beta}}_j = (\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\boldsymbol{y}_j$ with covariance $cov(\hat{\boldsymbol{\beta}}_j) = \sigma^2(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}$. The estimator is therefore normally distributed and can be written as

$$\hat{\boldsymbol{\beta}}_j = \boldsymbol{\beta}_j + \boldsymbol{r}_j,$$

where the residuals $\boldsymbol{r}_j \sim N(0, \sigma^2(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1})$.

In a hierarchical model, the 2-level model for the random coeffcients $\boldsymbol{\beta}_j$ is

$$\boldsymbol{\beta}_j = \boldsymbol{\gamma}'\boldsymbol{w}_j + \boldsymbol{\zeta}_j, j = 1, ..., K$$

where $\boldsymbol{w}_j$ are cluster-level covariates and the 2-level residuals are $\boldsymbol{\zeta}_j \sim N(0, \boldsymbol{\Psi})$. Find the variance of the hierarchical model estimator $\tilde{\boldsymbol{\beta}}_j$.

Hint: Write the model in reduced form and 'collect' all variation related to the estimation of $\boldsymbol{\beta}_j$.

4. The weighted empirical Bayes estimator of the regression parameter vector in the above linear hierarchical model is

$$\boldsymbol{\beta}_j^{EB} = \omega_j \hat{\boldsymbol{\beta}}_j + (1 - \omega_j)\boldsymbol{w}_j \hat{\boldsymbol{\gamma}},$$

where $\hat{\boldsymbol{\beta}}_j$ is the ordinary 'non-hierarchical' least-squares estimator and $\boldsymbol{w}_j \hat{\boldsymbol{\gamma}}$ the hierarchical model estimator of the model in Q3.

Consider the random intercept model with one covariate.
a) Write down the empirical Bayes estimator for random intercepts.
b) What is the form of the weight ω_j (or 1- ω_j) and its interpretation?
c) How would aggregation (use of cluster means only) affect the EB regression coefficients? How would 'non-hierarchical' modelling (neglecting clustering) affect them?

5. Use of latent variables in hierarchical modelling

6. Basic principles in estimating hierarchical models