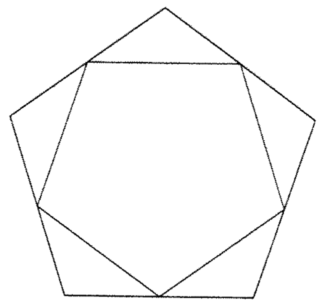
Geometry

1. Mid-Term Exam

25.2.2008

1. When the mid-points of the adjacent sides of a regular pentagon are connected by line segments, we obtain an inscribed regular pentagon as in the figure on the right. We denote by s the ratio of the length of the side of a regular pentagon to the length of the side of the inscribed regular pentagon.



(a) Give an expression for s in terms of trigonometric functions.

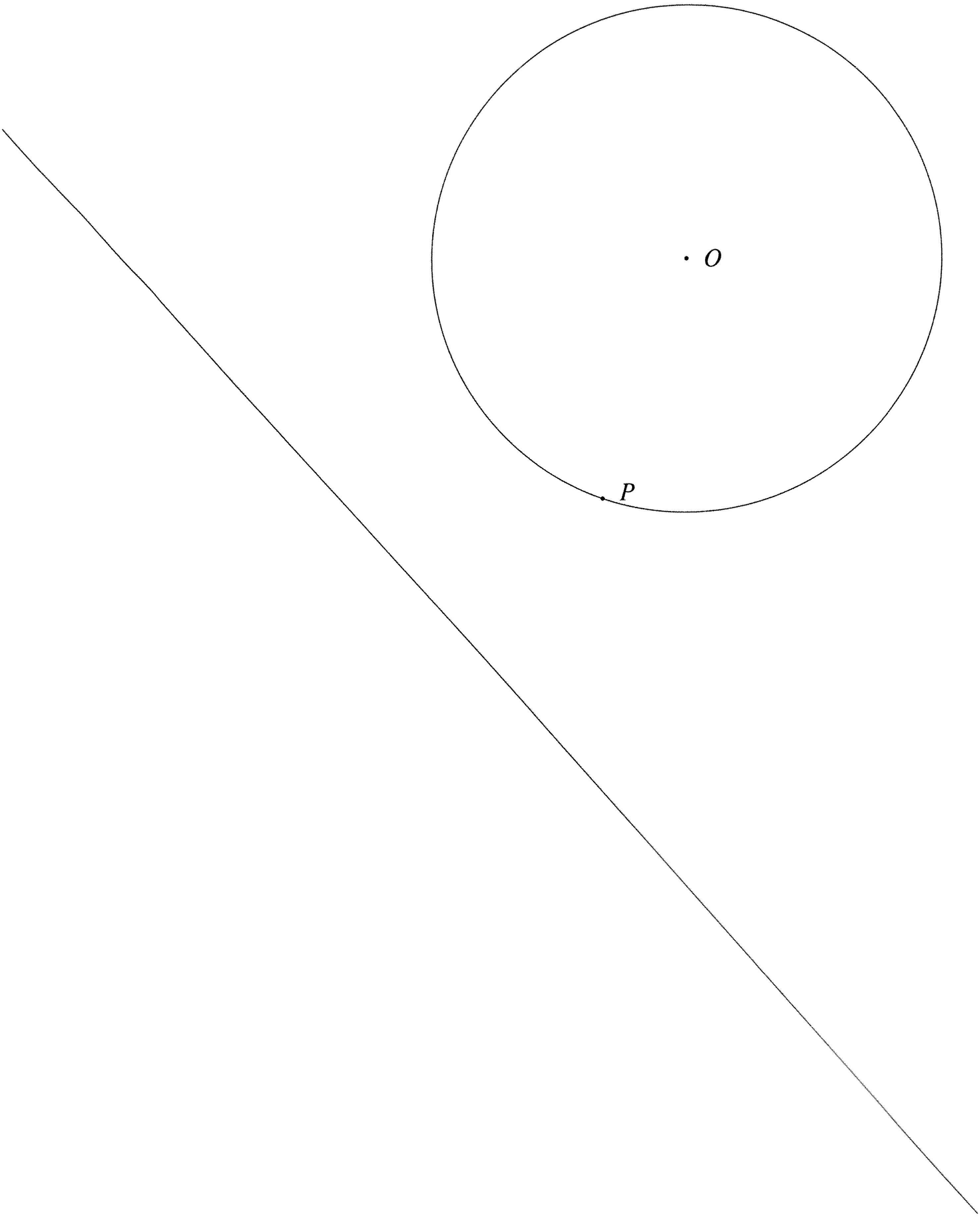
(b) Give an expression for s without trigonometric functions; however, the expression may contain the "golden number" ϕ.

2. The magnitudes of the angles of a triangle are α, β and γ. Let P, Q and R be the points at which the incircle of the triangle touches the sides of the triangle. Determine the magnitudes of the angles of the triangle PQR.

3. On the back side of this page you can see a line, a circle, the mid-point O of the circle and a point P on the circle. Use a straight-edge and a compass to construct a circle which touches the given line at some point and the circle at the point P. Give also a short explanation and motivation for your construction.
(If you wish, you can draw the figures on a separate sheet of paper.)

4. Write a PostScript program, which draws a figure like that seen above, where the side of the outer pentagon is approximately 10 cm.

[*Hint for Problem 3*: The mid-point of the circle must be equally far from the given line and the line touching the circle at P.]


O
P

Summary of some PostScript commands

The symbol $\emptyset$ means no arguments, or no return value.

1. Mathematical functions

Arguments	Command	Left on stack; side effects
$x\ y$	`add`	$x+y$
$x\ y$	`sub`	$x-y$
$x\ y$	`mul`	xy
$x\ y$	`div`	x/y
$x\ y$	`idiv`	the integral part of x/y
$x\ y$	`mod`	the remainder of x after division by y
x	`abs`	the absolute value of x
x	`neg`	$-x$
x	`ceiling`	the integer just above x
x	`floor`	the integer just below x
x	`round`	x rounded to nearest integer
x	`truncate`	x with fractional part chopped off
x	`sqrt`	square root of x
$x\ y$	`atan`	the polar argument of the point (x, y)
x	`cos`	$\cos x$ (x in degrees)
x	`sin`	$\sin x$ (x in degrees)
$x\ y$	`exp`	x^y
x	`ln`	$\ln x$
x	`log`	$\log x$ (base 10)

2. Stack operations

x	`pop`	$\emptyset$
$x\ y$	`exch`	$y\ x$
x	`dup`	$x\ x$
$x_0 \ldots x_{n-1}\ n\ i$	`roll`	$x_i \ldots x_0 \ldots x_{i-1}\ ni$

3. Dictionaries

name item	`def`	makes an entry in the current dictionary
n	`dict`	puts a dictionary of n null entries on the stack
dictionary d	`begin`	opens d for use
	`end`	closes the last dictionary opened

4. Conditionals

The first few return 'boolean' constants `true` or `false`. A few others have boolean values as arguments.

$x\ y$	`eq`	$x = y$?
$x\ y$	`ne`	$x \neq y$?
$x\ y$	`ge`	$x \geq y$?
$x\ y$	`gt`	$x > y$?
$x\ y$	`ge`	$x \geq y$?
$x\ y$	`gt`	$x > y$?
$x\ y$	`le`	$x \leq y$?
$x\ y$	`lt`	$x < y$?
$s\ t$	`and`	s and t are both true?
$s\ t$	`or`	at least one of s and t is true?
s	`not`	s is not true?
$s\ \{\ldots\}$	`if`	executes the procedure if s is true
$s\ \{\ldots\}\{\ldots\}$	`ifelse`	executes the first procedure if s is true, otherwise the second

5. Graphics state

$\emptyset$	`gsave`	saves the current graphics state, installs a new copy of it
$\emptyset$	`grestore`	brings back the last graphics state
x	`setlinewidth`	sets current linewidth to x (in current units)
x	`setlinecap`	determines how lines are capped
x	`setlinejoin`	determines how lines are joined
$[\ldots]\ x$	`setdash`	sets current dash pattern
g	`setgray`	sets current colour to a shade of grey
$r\ g\ b$	`setrgbcolor`	sets current colour

6. Coordinates

Here, a matrix is an array of 6 numbers. The CTM is the **C**urrent **T**ransformation **M**atrix.

$\emptyset$	`matrix`	puts a matrix on the stack
matrix m	`defaultmatrix`	fills m with the default TM, leaves it on the stack
m	`currentmatrix`	fills the matrix with the current CTM, leaves it
$x\ y$	`translate`	translates the origin by $[x, y]$
$a\ b$	`scale`	scales x by a, y by b
m	`concat`	multiplies the CTM by m
$m_1\ m_2\ m_3$	`concatmatrix`	fills m_3 with the matrix product $m_1 m_2$
$x\ y$	`transform`	$x'\ y'$, transform of $x\ y$ by the CTM
$x\ y\ m$	`transform`	$x'\ y'$, transform of $x\ y$ by m
$x\ y$	`itransform`	$x'\ y'$, transform of $x\ y$ by the inverse of the CTM
$x\ y\ m$	`itransform`	$x'\ y'$, transform of $x\ y$ by the inverse of m
$m_1\ m_2$	`invertmatrix`	m_2 (the matrix m_2 is filled by the inverse of m_1

7. Drawing

$\emptyset$	`newpath`	starts a new path, deleting the old one
$\emptyset$	`currentpoint`	the current point $x\ y$ in device coordinates
$x\ y$	`moveto`	begins a new piece of the current path
$x\ y$	`lineto`	adds a line to the current path
$x\ y$	`rmoveto`	relative move
$x\ y$	`rlineto`	relative line
$x\ y\ r\ a\ b$	`arc`	adds an arc from a to b, centred at (x, y), of radius r
$x\ y\ r\ a\ b$	`arcn`	negative direction
$x_1\ y_1\ x_2\ y_2\ x_3\ y_3$	`curveto`	adds a Bezier curve to the current path
$\emptyset$	`closepath`	closes up the current path back to the last point moved to
$\emptyset$	`stroke`	draws the current path
$\emptyset$	`fill`	fills the outline made by the current path
$\emptyset$	`clip`	clips drawing to the region outlined by the current path

Department of Mathematics and Statistics
Geometry
1. exam
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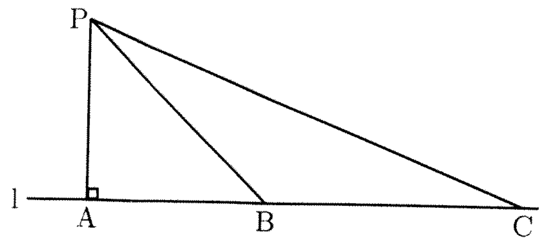
(1) **Elementa.** To draw a straight line at right angles to a given straight line from a given point on it.

Hint: You can use Postulates P1–P4 ja Propositions I.1–I.10.

(2) **Absolutely geometry.**



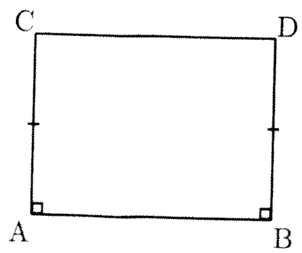
Let l be a straight line and P a point not on it. Draw a perpendicular PA to l. In the line l we set $AB = AP$ and $BC = BP$. Show that $\angle PCA \leq R/4$.

Hint: You can use Postulates P1–P4 ja Propositions I.1–I.28 and the first theorem of Saccher and Legendre.

(3) **Hilbert's axioms.** Let l be a straight line. Show that there exist points P and R, which are not on l and which are on opposite sides of l.

Hint: You can use only Hilbert's axioms.

(4) **Hyperbolic geometry.**



Let ABCD a Saccher quadrilateral: angles A and B right angles and AC=BD. Show in hyperbolic geometry that angles C and D are congruent to each other and they are acute angles.

Hint: You can use Postulates P1–P4, Propositions I.1–I.28, hyperbolic parallel postulate and the fact what is the sum of angles in a triangle.

Postulate 1.
To draw a straight line from any point to any point.

Postulate 2.
To produce a finite straight line continuously in a straight line.

Postulate 3.
To describe a circle with any center and radius.

Postulate 4.
That all right angles equal one another.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Proposition 1.
To construct an equilateral triangle on a given finite straight line.

Proposition 2.
To place a straight line equal to a given straight line with one end at a given point.

Proposition 3.
To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition 4.
If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proposition 5.
In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition 6.
If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition 7.
Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition 8.
If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition 9.
To bisect a given rectilinear angle.

Proposition 10.
To bisect a given finite straight line.

Proposition 11.
To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12.
To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition 13.
If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition 14.
If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition 15.
If two straight lines cut one another, then they make the vertical angles equal to one another.
Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.

Proposition 16.
In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Proposition 17.
In any triangle the sum of any two angles is less than two right angles.

Proposition 18.
In any triangle the angle opposite the greater side is greater.

Proposition 19.
In any triangle the side opposite the greater angle is greater.

Proposition 20.
In any triangle the sum of any two sides is greater than the remaining one.

Proposition 21.
If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition 22.
To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition 23.
To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition 24.
If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition 25.
If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

Proposition 26.
If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 27.
If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28.
If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Betweenness Axioms

AXIOM B-1. If A * B * C, then A, B, and C are three distinct points all lying on the same line, and C * B * A.

AXIOM B-2. Given any two distinct points B and D, there exist points A, C and E lying on $\overleftrightarrow{BD}$ such that A * B * D, B * C * D and B * D * E.

AXIOM B-3. If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

AXIOM B-4. For every line *l* and for any three points A, B, and C not lying on *l*:

(i) if A and B are on the same side of *l*, and B and C are on the same side of *l*, then A and C are on the same side of *l*.

(ii) if A and B are on opposite sides of *l* and B and C are on oppo sides of *l*, then A and C are on the same side of *l*.

Incidence Axioms

AXIOM I-1. For every point P and for every point Q not equal to P there exists a unique line *l* that passes through P and Q.

AXIOM I-2. For every line *l* there exist at least two distinct points that are incident with *l*.

AXIOM I-3. There exist three distinct points with the property that no line is incident with all three of them.