Final Examination

June 18, 2004

Note: The exam will be graded on a maximum of points of 24. Each of the five problems below yields 6 points if answered completely, these 6 points being distributed among subproblems according to difficulty. A partial answer to a subproblem may yield points.

Problem 1

Find all fixed points of the following maps of $\mathbb{R}^2$. For each fixed point, determine if it is hyperbolic or nonhyperbolic, and, in the former case, if it is a source, a sink, or a saddle node.

(1) $f(x, y) = (\exp(x) - 1, \frac{1}{2} y \cos(y))$

(2) $g(x, y) = (\sin(\pi y), \frac{1}{3} x)$

(3) $h(x, y) = (2x, xy - \frac{1}{3} y - 13x^2)$

Problem 2

Consider the (local) diffeomorphism h at the origin given in Problem 1.3.

(1) Check that the stable manifold $W^s((0,0))$ is given by $\{(0, y) \mid y \in \mathbb{R}\}$.

(2) Why can the local unstable manifold $W^u_{\text{loc}}((0,0))$ be expressed as $\{(x, s(x)) \mid x \in \mathcal{U}_0\}$ for $\mathcal{U}_0 \subseteq \mathbb{R}$ some neighborhood of the origin and $s : \mathcal{U}_0 \to \mathbb{R}$ a function satisfying $s(0) = 0$ and $s'(0) = 0$?

(3) Find the equation that must be satisfied by the function s.

(4) Expressing s as a power series $s(x) = \sum_{k \geq 2} s_k x^k$, determine the coefficients s_k, $k \geq 2$. Draw a (qualitative) picture of the phase portrait of h near the origin.

Problem 3

With $\mu \in [0, 4]$ and f_μ the logistic map, let g_μ be the map on the interval $[0, 1]$ defined by

$$g_\mu(x) = \begin{cases} 1 - x & \text{if } x \in [0, \frac{2}{3}], \\ \frac{1}{3} - \frac{1}{3} f_\mu(3x - 2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

(1) Prove that $g^2_\mu : [0, 1/3] \to [0, 1/3]$ is topologically conjugate to $f_\mu : [0, 1] \to [0, 1]$.

(2) Prove that g_μ has a periodic point of prime period k in $[0, 1/3]$ if and only if k is even and f_μ has a periodic point of prime period $k/2$. Prove that two corresponding

periodic points for g_μ and f_μ have same stability. What does this imply regarding the structure of periodic points of g_μ as μ increases from zero ?

(3) Prove that the set $[0, 1/3] \cup [2/3, 1]$ is a chaotic attractor for g_4. (Recall that a set J is a chaotic attractor for a map $f : J \to J$ if f is chaotic on J and if J is an attractor for f.)

Problem 4

Consider the continuous-time dynamical system in $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{x} &= 2\cos(x) - \cos(y) \\ \dot{y} &= 2\cos(y) - \cos(x) \end{aligned} \tag{4.1}$$

By periodicity of the vector field, it is enough to study (4.1) in $D \equiv [-\pi, \pi] \times [-\pi, \pi]$. For points (3) and (4) below, you may use the following *Result*: A periodic orbit of a planar system always contains at least one fixed point in its interior.

(1) Find the fixed points of (4.1) in D and determine their stability.

(2) Check that the system has the following symmetries: If $(x(t), y(t))$ is a solution of (4.1), then $(y(t), x(t))$ and $(-x(-t), -y(-t))$ are both solutions of (4.1).

(3) Prove that there is a heteroclinic connection from the saddle node in $[0, \pi] \times [-\pi, 0]$ to the sink in $[0, \pi] \times [0, \pi]$. *Hint:* prove first that the triangle with vertices $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a trapping region.

(4) Using the symmetries of point (2), prove that D contains no periodic orbit and draw the phase portrait of (4.1) in D.

Problem 5

Let $f_{\gamma,\lambda} : \mathbb{R} \to \mathbb{R}$ be the two-parameters family defined for $\gamma > 0$ and $\lambda \in \mathbb{R}$ by

$$f_{\gamma,\lambda}(x) = \gamma(x^3 - x) + \lambda \,.$$

Recall: Nondegeneracy conditions for the saddle-node bifurcation: $\frac{\partial}{\partial \lambda} f_\lambda(p_0)|_{\lambda=\lambda_0} \neq 0$ and $f''_{\lambda_0}(p_0) \neq 0$. Nondegeneracy condition for the period-doubling bifurcation: $[2\frac{\partial}{\partial \lambda} f'_\lambda(p_0) + (\frac{\partial}{\partial \lambda} f_\lambda)(p_0) f''_\lambda(p_0)]_{\lambda=\lambda_0} \neq 0$.

(1) For all $\gamma > 0$ fixed, find the complete set of nondegenerate fixed point bifurcations that occur in the family $f_{\gamma,\lambda}$ as λ varies in $\mathbb{R}$. Justify your answer by invoking the appropriate theorems. *Hint:* determine first the solutions of $f'_{\gamma,\lambda}(x) = \pm 1$.

(2) Draw (qualitative) pictures of the bifurcation diagrams in the (λ, x)-plane for typical values of γ. Indicate the stability of each fixed point branch.

(3) Let $\lambda = 0$. Prove that for γ large enough, there exists an interval I such that I contains no fixed point and $f^3_{\gamma,0}(I) \supset I$. Conclude that $f_{\gamma,0}$ has periodic points of all periods. What does this result suggest about the bifurcation diagrams in (2) ?