Department of Mathematics and Statistics
Introduction to differential geometry
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Final exam
20.1.2005

1. Prove the following *(Banach's) fixed point theorem:* Let X be a complete metric space and let $f\colon X \to X$ be a mapping. Suppose that there exists a constant $L \in \mathbb{R}$, $0 \leq L < 1$, such that

$$d(f(x), f(y)) \leq Ld(x, y)$$

for all $x, y \in X$. Then there exists exactly one point $x_0 \in X$ such that $f(x_0) = x_0$.

2. Let M, N, and L be differentiable manifolds and let $f\colon M \to N$ and $g\colon N \to L$ be smooth mappings. Prove that

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$$

for all $p \in M$.

3. Let $\mathcal{D} = \mathbb{R} \times \mathbb{R}^2$ and $\theta\colon \mathcal{D} \to \mathbb{R}^2$,

$$\theta\big(t, (x, y)\big) = \theta_t(x, y) = \big(xe^{2t}, ye^{-3t}\big).$$

Prove that θ is a flow and find its infinitesimal generator.

4. Let $\omega \in \mathcal{A}^1(M)$ be a smooth differential 1-form and let $X \in \mathcal{T}(M)$ and $Y \in \mathcal{T}(M)$ be smooth vector fields. Prove that

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big).$$

5. Let $X \in \mathcal{T}(M)$ and let ω and η be smooth differential forms. Prove that

$$L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta).$$