Differential equations I

Midterm exam, 18.10.2005

No cellphones or calculators, MAOL tables or equivalents are allowed

1. Determine the solution of the initial value problem

$$y' = y \sin x, \quad y(0) = -1.$$

2. (From the homework) Show that the equation

$$y^2 + 2xy - x^2 y' = 0$$

is not exact. Using a suitable power of x or y as an integrating factor transform this into an exact equation, and then solve the exact version.

A helpful fact: An integral function $F(x, y)$ for the exact equation $M(x, y) + N(x, y)y' = 0$ is given by

$$F(x, y) = \int_{x_0}^{x} M(t, y)\, dt + \int N(x_0, y)\, dy.$$

3. Solve the initial value problem

$$y' + 4y - e^{-x} = 0, \quad y(0) = 4/3.$$

4. (a) Do functions $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$ form a fundamental system for the equation

$$y'' - 2y' + 2y = 0?$$

Give proof of your claim.

(b) Determine the general solution of the equation

$$y'' - 5y' + 6y = 0.$$

Differential equations I

Final exam 16.11.2005

No calculators or cell phones, only math tables are allowed

1. Solve the initial value problem

$$y' = y^2 \sin x \cos x, \quad y(0) = 2.$$

2. Determine the general solution of the equation

$$4x + 18y + 6xy' = 0.$$

Hint: use a suitable integrating factor !

3. Solve the initial value problem

$$y' + 3y/x + 2 = x^2, \quad y(1) = 1.$$

4. Solve the equation

$$y' + 6y = -2xy^5.$$

5. Solve the initial value problem

$$y'' - 3y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Differential equations I

Final Exam, 16.11. 2006

1. Solve the initial value problem

$$\begin{aligned} \dot{x}(t) &= x(t)^2 \cos t, \\ x(0) &= \frac{1}{2}. \end{aligned}$$

2. Determine the general solution to the equation

$$\ddot{x}(t) - 2\dot{x}(t) + x(t) = e^t.$$

3. Show that the differential equation

$$2x\cos^2 y + \left(2y - x^2 \sin 2y\right) \frac{dy}{dx} = 0$$

is exact and determine its general solution. It suffices to give the solution in implicit form.

4. Determine the general solution to Bernoulli's equation

$$y'(x) + 2xy(x) + xy(x)^4 = 0.$$

5. Consider the SIR-model

$$\begin{aligned} \frac{dS}{dt} &= -\beta S I, \\ \frac{dI}{dt} &= \beta S I - \alpha I, \\ \frac{dR}{dt} &= \alpha I \end{aligned}$$

in the set $S \geq 0$, $I \geq 0$, $R \geq 0$. Here $\alpha > 0$, $\beta > 0$.

(a) Show that $N := S + I + R$ is constant.

(b) Show that if $R_0 := \frac{\beta N}{\alpha} > 1$, then the limit $s_\infty := \lim_{t\to\infty} \frac{S(t)}{N}$ exists and satisfies the equation

$$s_\infty = 1 + \frac{1}{R_0} \ln s_\infty.$$