Course in probability, exam 9.8.07
Lecturer: Jukka Corander

Justify your answers properly in every problem. Each problem is worth of 6 points.

1. Let A and B be independent random events. Show that then the events A^c and B^c are independent. (A^c and B^c refer to the complementary events of A and B.)

2. On the grounds of a certain test, your friend has been diagnosed as having a disease which is very rare in Finland, occurring with a frequency of one in a million. The diagnosis is based on a laboratory test provided by an international drug company. Experiments have shown that the test has the following accuracy: 1 % of people without the disease are indicated as having it by the test ("false positive rate"), and 1 % of diseased people are not indicated as having it ("false negative rate"). Given this information, what is the probability that your friend really has the disease? Hint: use Bayes' theorem.

3. Assume that a continuous random variable X is distributed according to $N(0,1)$, the normal distribution with expectation 0 and variance 1. Define a random variable Y by $Y = aX + b$, where a and b are real numbers with $a > 0$, $-\infty < b < \infty$. Derive the probability density function of Y.

4. Use a Venn diagram to represent two random events A and B that are conditionally independent given a third random event C. Note that in a Venn diagram the areas of the events are in proportion to their probabilities.

5. Random variables X and Y have a joint probability distribution described by the density function

$$f(x,y) = \begin{cases} \frac{5x^2}{2}, & -1 \le x \le 1,\ 0 \le y \le x^2 \\ 0, & \text{otherwise.} \end{cases}$$

   Determine the conditional density function $f(y|X=x)$ for Y and calculate the conditional expectation $E(Y|X=x)$.

Course in probability (todennäköisyyslaskennan kurssi)
Exam 24. 1. 2008

Using table books is not allowed!

1. A gambler has two coins in his pocket. One is a normal coin, with heads on one side and tails on the other. The other coin is special, having two heads. The gambler selects one coin at random and tosses it, getting heads. What is the probability that the other side of this coin is also heads? Use Bayes' rule.

2. a) What is the definition of the *Poisson*(μ) distribution? What are its mean and variance? (No proofs are required.)
   b) The number of chocolate chips in a certain type of cookie has a Poisson distribution. The producer of the cookies wants that a randomly chosen cookie has at least one chocolate chip with a probability ≥ 0.99. How should the parameter μ be chosen? What is the probability that the cookie has at least two chocolate chips?

3. a) What is the definition of the joint cumulative distribution function (cdf) of a random vector (X, Y)? In the case when (X, Y) has a continuous distribution, how is the joint probability density function (pdf) derived from the joint cdf?
   b) Prove that the function $F\colon \mathbb{R}^2 \to \mathbb{R}$,
   $$F(x,y) = \begin{cases} 1 & \text{for } y \geq -x, \\ 0 & \text{for } y < -x, \end{cases}$$
   is not the joint cdf of any bivariate random vector.

4. Let $X_1, X_2, \ldots$ be a sequence of iid random variables with a mean μ and a finite variance σ^2. According to the Central Limit Theorem, the sums (or averages) of the variables $X_1, \ldots, X_n$, under suitable normalization, converge in distribution to the standard normal distribution. State this result in its exact form. Use it to approximate the probability $P\{\sum_{i=1}^{24} X_i > 10\}$ when $X_1, \ldots, X_{24}$ are independent and uniformly distributed over the interval $[0, 1]$.

   See overleaf for a table of the cdf of the standard normal distribution.

5. Assume that U and V are independent random variables and have the uniform distribution over $[0, 1]$. Calculate the joint probability density function (pdf) of the random vector (U, V). Define $X = U$ ja $Y = UV$. Derive the joint pdf of the random vector (X, Y). In particular, specify clearly in which set it equals zero and in which set it is positive.

Table of the Standard Normal Distribution Function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}u^2\right) du$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
0.04	0.5160	0.64	0.7389	1.24	0.8925	1.84	0.9671	2.44	0.9927
0.05	0.5199	0.65	0.7422	1.25	0.8944	1.85	0.9678	2.45	0.9929
0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931
0.07	0.5279	0.67	0.7486	1.27	0.8980	1.87	0.9693	2.47	0.9932
0.08	0.5319	0.68	0.7517	1.28	0.8997	1.88	0.9699	2.48	0.9934
0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945
0.13	0.5517	0.73	0.7673	1.33	0.9082	1.93	0.9732	2.56	0.9948
0.14	0.5557	0.74	0.7704	1.34	0.9099	1.94	0.9738	2.58	0.9951
0.15	0.5596	0.75	0.7734	1.35	0.9115	1.95	0.9744	2.60	0.9953
0.16	0.5636	0.76	0.7764	1.36	0.9131	1.96	0.9750	2.62	0.9956
0.17	0.5675	0.77	0.7794	1.37	0.9147	1.97	0.9756	2.64	0.9959
0.18	0.5714	0.78	0.7823	1.38	0.9162	1.98	0.9761	2.66	0.9961
0.19	0.5753	0.79	0.7852	1.39	0.9177	1.99	0.9767	2.68	0.9963
0.20	0.5793	0.80	0.7881	1.40	0.9192	2.00	0.9773	2.70	0.9965
0.21	0.5832	0.81	0.7910	1.41	0.9207	2.01	0.9778	2.72	0.9967
0.22	0.5871	0.82	0.7939	1.42	0.9222	2.02	0.9783	2.74	0.9969
0.23	0.5910	0.83	0.7967	1.43	0.9236	2.03	0.9788	2.76	0.9971
0.24	0.5948	0.84	0.7995	1.44	0.9251	2.04	0.9793	2.78	0.9973
0.25	0.5987	0.85	0.8023	1.45	0.9265	2.05	0.9798	2.80	0.9974
0.26	0.6026	0.86	0.8051	1.46	0.9279	2.06	0.9803	2.82	0.9976
0.27	0.6064	0.87	0.8079	1.47	0.9292	2.07	0.9808	2.84	0.9977
0.28	0.6103	0.88	0.8106	1.48	0.9306	2.08	0.9812	2.86	0.9979
0.29	0.6141	0.89	0.8133	1.49	0.9319	2.09	0.9817	2.88	0.9980
0.30	0.6179	0.90	0.8159	1.50	0.9332	2.10	0.9821	2.90	0.9981
0.31	0.6217	0.91	0.8186	1.51	0.9345	2.11	0.9826	2.92	0.9983
0.32	0.6255	0.92	0.8212	1.52	0.9357	2.12	0.9830	2.94	0.9984
0.33	0.6293	0.93	0.8238	1.53	0.9370	2.13	0.9834	2.96	0.9985
0.34	0.6331	0.94	0.8264	1.54	0.9382	2.14	0.9838	2.98	0.9986
0.35	0.6368	0.95	0.8289	1.55	0.9394	2.15	0.9842	3.00	0.9987
0.36	0.6406	0.96	0.8315	1.56	0.9406	2.16	0.9846	3.05	0.9989
0.37	0.6443	0.97	0.8340	1.57	0.9418	2.17	0.9850	3.10	0.9990
0.38	0.6480	0.98	0.8365	1.58	0.9429	2.18	0.9854	3.15	0.9992
0.39	0.6517	0.99	0.8389	1.59	0.9441	2.19	0.9857	3.20	0.9993
0.40	0.6554	1.00	0.8413	1.60	0.9452	2.20	0.9861	3.25	0.9994
0.41	0.6591	1.01	0.8437	1.61	0.9463	2.21	0.9864	3.30	0.9995
0.42	0.6628	1.02	0.8461	1.62	0.9474	2.22	0.9868	3.35	0.9996
0.43	0.6664	1.03	0.8485	1.63	0.9485	2.23	0.9871	3.40	0.9997
0.44	0.6700	1.04	0.8508	1.64	0.9495	2.24	0.9875	3.45	0.9997
0.45	0.6736	1.05	0.8531	1.65	0.9505	2.25	0.9878	3.50	0.9998
0.46	0.6772	1.06	0.8554	1.66	0.9515	2.26	0.9881	3.55	0.9998
0.47	0.6808	1.07	0.8577	1.67	0.9525	2.27	0.9884	3.60	0.9998
0.48	0.6844	1.08	0.8599	1.68	0.9535	2.28	0.9887	3.65	0.9999
0.49	0.6879	1.09	0.8621	1.69	0.9545	2.29	0.9890	3.70	0.9999
0.50	0.6915	1.10	0.8643	1.70	0.9554	2.30	0.9893	3.75	0.9999
0.51	0.6950	1.11	0.8665	1.71	0.9564	2.31	0.9896	3.80	0.9999
0.52	0.6985	1.12	0.8686	1.72	0.9573	2.32	0.9898	3.85	0.9999
0.53	0.7019	1.13	0.8708	1.73	0.9582	2.33	0.9901	3.90	1.0000
0.54	0.7054	1.14	0.8729	1.74	0.9591	2.34	0.9904	3.95	1.0000
0.55	0.7088	1.15	0.8749	1.75	0.9599	2.35	0.9906	4.00	1.0000
0.56	0.7123	1.16	0.8770	1.76	0.9608	2.36	0.9909		
0.57	0.7157	1.17	0.8790	1.77	0.9616	2.37	0.9911		
0.58	0.7190	1.18	0.8810	1.78	0.9625	2.38	0.9913		
0.59	0.7224	1.19	0.8830	1.79	0.9633	2.39	0.9916		

Course in probability (todennäköisyyslaskennan kurssi)
Exam 3. 4. 2008

Using table books is not allowed!

1. A question in a multiple choice exam has four alternative answers. The student knows the correct answer with the probability 0.8. If the student doesn't know the answer, he/she makes a random guess, thus getting the correct answer with the probability 1/4. What is the probability that the student *knew* the correct answer if his/her answer is correct? Use Bayes' Rule.

2. a) What is the definition of the binomial distribution binomial(n, θ)? What are its mean and variance? (No proofs are needed.)
   b) Give an example of a concrete random phenomenon or experiment and a related random variable which has the binomial$(10, \frac{1}{2})$ distribution.

3. Bob commutes from home to work and back by subway which runs at the interval of 5 minutes. When going to the subway station, he doesn't pay any attention to the time, so we may assume that his waiting times in the morning and in the evening are each uniformly distributed. How is Bob's combined waiting time (in one day) distributed? What are its mean and standard deviation?

4. Assume that U ja V are independent random variables, each having the standard normal distribution n$(0, 1)$. Define $X = U + V + 1$ and $Y = U - 2V$. Determine the distributions of X and Y and calculate the correlation coefficient ρ_{XY}.

5. a) Let $Y_1, Y_2, \ldots$ be a sequence of random variables. What is the definition of this sequence converging in probability to a random variable Y?
   b) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and variance $\sigma^2 < \infty$. Define $\overline{X}_n = \sum_{i=1}^{n} X_i/n$. Prove the Weak Law of Large Numbers, according to which the variables $\overline{X}_n$ converge in probability to μ.

Course in probability (todennäköisyyslaskennan kurssi)
Exam 12. 6. 2008

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1. A question in a multiple choice exam has four alternative answers. The student knows the correct answer with the probability 0.8. If the student doesn't know the answer, he/she makes a random guess, thus getting the correct answer with the probability 1/4. What is the probability that the student *knew* the correct answer if his/her answer is correct? Use Bayes' Rule.

2. a) What is the definition of the binomial distribution binomial(n, θ)? What are its mean and variance? (No proofs are needed.)
   b) Give an example of a concrete random phenomenon or experiment and a related random variable which has the binomial$(10, \frac{1}{2})$ distribution.

3. Bob commutes from home to work and back by subway which runs at the interval of 5 minutes. When going to the subway station, he doesn't pay any attention to the time, so we may assume that his waiting times in the morning and in the evening are each uniformly distributed. How is Bob's combined waiting time (in one day) distributed? What are its mean and standard deviation?

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Course in probability (todennäköisyyslaskennan kurssi)
Exam 14. 8. 2008

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1. A question in a multiple choice exam has four alternative answers. The student knows the correct answer with the probability 0.8. If the student doesn't know the answer, he/she makes a random guess, thus getting the correct answer with the probability 1/4. What is the probability that the student *knew* the correct answer if his/her answer is correct? Use Bayes' Rule.

2. a) What is the definition of the binomial distribution binomial(n, θ)? What are its mean and variance? (No proofs are needed.)
   b) Give an example of a concrete random phenomenon or experiment and a related random variable which has the binomial$(10, \frac{1}{2})$ distribution.

3. Bob commutes from home to work and back by subway which runs at the interval of 5 minutes. When going to the subway station, he doesn't pay any attention to the time, so we may assume that his waiting times in the morning and in the evening are each uniformly distributed. How is Bob's combined waiting time (in one day) distributed? What are its mean and standard deviation?

4. Assume that U ja V are independent random variables, each having the standard normal distribution $\mathrm{n}(0, 1)$. Define $X = U + V + 1$ and $Y = U - 2V$. Determine the distributions of X and Y and calculate the correlation coefficient ρ_{XY}.

5. a) Let $Y_1, Y_2, \ldots$ be a sequence of random variables. What is the definition of this sequence converging in probability to a random variable Y?
   b) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and variance $\sigma^2 < \infty$. Define $\overline{X}_n = \sum_{i=1}^n X_i/n$. Prove the Weak Law of Large Numbers, according to which the variables $\overline{X}_n$ converge in probability to μ.

Table of the Standard Normal Distribution Function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}u^2\right) du$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
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0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
0.04	0.5160	0.64	0.7389	1.24	0.8925	1.84	0.9671	2.44	0.9927
0.05	0.5199	0.65	0.7422	1.25	0.8944	1.85	0.9678	2.45	0.9929
0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931
0.07	0.5279	0.67	0.7486	1.27	0.8980	1.87	0.9693	2.47	0.9932
0.08	0.5319	0.68	0.7517	1.28	0.8997	1.88	0.9699	2.48	0.9934
0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945
0.13	0.5517	0.73	0.7673	1.33	0.9082	1.93	0.9732	2.56	0.9948
0.14	0.5557	0.74	0.7704	1.34	0.9099	1.94	0.9738	2.58	0.9951
0.15	0.5596	0.75	0.7734	1.35	0.9115	1.95	0.9744	2.60	0.9953
0.16	0.5636	0.76	0.7764	1.36	0.9131	1.96	0.9750	2.62	0.9956
0.17	0.5675	0.77	0.7794	1.37	0.9147	1.97	0.9756	2.64	0.9959
0.18	0.5714	0.78	0.7823	1.38	0.9162	1.98	0.9761	2.66	0.9961
0.19	0.5753	0.79	0.7852	1.39	0.9177	1.99	0.9767	2.68	0.9963
0.20	0.5793	0.80	0.7881	1.40	0.9192	2.00	0.9773	2.70	0.9965
0.21	0.5832	0.81	0.7910	1.41	0.9207	2.01	0.9778	2.72	0.9967
0.22	0.5871	0.82	0.7939	1.42	0.9222	2.02	0.9783	2.74	0.9969
0.23	0.5910	0.83	0.7967	1.43	0.9236	2.03	0.9788	2.76	0.9971
0.24	0.5948	0.84	0.7995	1.44	0.9251	2.04	0.9793	2.78	0.9973
0.25	0.5987	0.85	0.8023	1.45	0.9265	2.05	0.9798	2.80	0.9974
0.26	0.6026	0.86	0.8051	1.46	0.9279	2.06	0.9803	2.82	0.9976
0.27	0.6064	0.87	0.8079	1.47	0.9292	2.07	0.9808	2.84	0.9977
0.28	0.6103	0.88	0.8106	1.48	0.9306	2.08	0.9812	2.86	0.9979
0.29	0.6141	0.89	0.8133	1.49	0.9319	2.09	0.9817	2.88	0.9980
0.30	0.6179	0.90	0.8159	1.50	0.9332	2.10	0.9821	2.90	0.9981
0.31	0.6217	0.91	0.8186	1.51	0.9345	2.11	0.9826	2.92	0.9983
0.32	0.6255	0.92	0.8212	1.52	0.9357	2.12	0.9830	2.94	0.9984
0.33	0.6293	0.93	0.8238	1.53	0.9370	2.13	0.9834	2.96	0.9985
0.34	0.6331	0.94	0.8264	1.54	0.9382	2.14	0.9838	2.98	0.9986
0.35	0.6368	0.95	0.8289	1.55	0.9394	2.15	0.9842	3.00	0.9987
0.36	0.6406	0.96	0.8315	1.56	0.9406	2.16	0.9846	3.05	0.9989
0.37	0.6443	0.97	0.8340	1.57	0.9418	2.17	0.9850	3.10	0.9990
0.38	0.6480	0.98	0.8365	1.58	0.9429	2.18	0.9854	3.15	0.9992
0.39	0.6517	0.99	0.8389	1.59	0.9441	2.19	0.9857	3.20	0.9993
0.40	0.6554	1.00	0.8413	1.60	0.9452	2.20	0.9861	3.25	0.9994
0.41	0.6591	1.01	0.8437	1.61	0.9463	2.21	0.9864	3.30	0.9995
0.42	0.6628	1.02	0.8461	1.62	0.9474	2.22	0.9868	3.35	0.9996
0.43	0.6664	1.03	0.8485	1.63	0.9485	2.23	0.9871	3.40	0.9997
0.44	0.6700	1.04	0.8508	1.64	0.9495	2.24	0.9875	3.45	0.9997
0.45	0.6736	1.05	0.8531	1.65	0.9505	2.25	0.9878	3.50	0.9998
0.46	0.6772	1.06	0.8554	1.66	0.9515	2.26	0.9881	3.55	0.9998
0.47	0.6808	1.07	0.8577	1.67	0.9525	2.27	0.9884	3.60	0.9998
0.48	0.6844	1.08	0.8599	1.68	0.9535	2.28	0.9887	3.65	0.9999
0.49	0.6879	1.09	0.8621	1.69	0.9545	2.29	0.9890	3.70	0.9999
0.50	0.6915	1.10	0.8643	1.70	0.9554	2.30	0.9893	3.75	0.9999
0.51	0.6950	1.11	0.8665	1.71	0.9564	2.31	0.9896	3.80	0.9999
0.52	0.6985	1.12	0.8686	1.72	0.9573	2.32	0.9898	3.85	0.9999
0.53	0.7019	1.13	0.8708	1.73	0.9582	2.33	0.9901	3.90	1.0000
0.54	0.7054	1.14	0.8729	1.74	0.9591	2.34	0.9904	3.95	1.0000
0.55	0.7088	1.15	0.8749	1.75	0.9599	2.35	0.9906	4.00	1.0000
0.56	0.7123	1.16	0.8770	1.76	0.9608	2.36	0.9909		
0.57	0.7157	1.17	0.8790	1.77	0.9616	2.37	0.9911		
0.58	0.7190	1.18	0.8810	1.78	0.9625	2.38	0.9913		
0.59	0.7224	1.19	0.8830	1.79	0.9633	2.39	0.9916		

Todennäköisyyslaskennan kurssi, final examination 11.6.2009 (based on Casella and Berger)

No mathematical tables are allowed in the examination.

1. Suppose $Y \sim N(\mu, \sigma^2)$, where $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2, and define $X = \exp(Y)$. Find the probability density function (pdf), expected value and variance of X.

2. Let random variables X and Y have the joint pdf

$$f(x, y) = c\,xy, \quad \text{when } 0 < y < x < 1,$$

and zero elsewhere.

a) Find the value of c.

b) Calculate the probability $P(X + Y < 1)$.

3. Let U and V be independent random variables with distributions

$$U \sim N(0, 1), \qquad V \sim N(1, 2^2),$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

a) Find the expected value and variance of $U - 2V + 3$.

b) Find $E \exp(U + V)$.

4. Consider the hierarchical model

$$X \mid Y \sim \text{Bin}(Y, p)$$
$$Y \sim \text{Poi}(\mu),$$

where $\mu > 0$ and $0 < p < 1$ are constants. Here $\text{Poi}(\mu)$ denotes the Poisson distribution with mean μ and $\text{Bin}(n, p)$ the binomial distribution with sample size parameter n and probability parameter p.

a) Write an expression for the joint probability mass function of X and Y.

b) Find $E(X \mid Y)$ and EX.

c) Find $\text{Var}(X \mid Y)$ and $\text{Var}\, X$.

5. Let X and Y be independent random variables with gamma distributions

$$X \sim \text{Gamma}(\alpha, 1), \qquad Y \sim \text{Gamma}(\beta, 1).$$

The pdf of the gamma distribution $G(\alpha, \beta)$ is

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\, \beta^\alpha}\, x^{\alpha - 1}\, e^{-x/\beta}, \qquad x > 0.$$

Find the pdf of $U = X/Y$.