Todennäköisyyslaskennan kurssi, fall 2008, 2nd course exam 15.12.2008

You are not allowed to use mathematical tables in the exam. Give feedback about the course in WebOodi.

1. Let the random vector (X, Y) be uniformly distributed within the circle having center $(\frac{1}{2}, 0)$ and radius $1/2$. (The area of this circle is $\pi/4$, and the joint pdf has nonzero value, when $(x - \frac{1}{2})^2 + y^2 < \frac{1}{4}$.) Find the marginal pdf of X. Find and recognize the conditional pdf of Y given $X = x$ (when $0 < x < 1$).

2. Consider the hierarchical model

$$X \mid S \sim N(\mu, \frac{1}{S})$$
$$S \sim \mathrm{Gam}(\alpha, 1),$$

where $\alpha > 1$ and $\mu \in \mathbb{R}$ are constants, and the second parameter of the normal distribution $N(\mu, \sigma^2)$ is its variance. For the formula of the pdf of the gamma distribution, see below.

Determine $E[X \mid S]$, EX (hint: iterated expectation) $E[X^2 \mid S]$, EX^2 and $E(SX^2)$.

3. Let X and Y be independent gamma distributed random variables such that

$$X \sim \mathrm{Gam}(\alpha, 1) \qquad Y \sim \mathrm{Gam}(\beta, 1).$$

(The pdf of the gamma distribution is given below.) Define

$$S = X + Y, \qquad Z = X/S.$$

Find the joint pdf of X and Z, and the marginal pdf of Z.

4. Let the random vector (X, Y) have the bivariate normal distribution with

$$EX = 0, \quad EY = 0, \quad \operatorname{var} X = \sigma_X^2, \quad \operatorname{var} Y = \sigma_Y^2, \quad \operatorname{cov}(X, Y) = \rho\,\sigma_X \sigma_Y,$$

where $\sigma_X, \sigma_Y > 0$ and $-1 < \rho < 1$. Define Z by $Z = Y - bX$, where the value of the constant b is chosen in part a of the problem

a) Determine the value of the constant b so that $\operatorname{cov}(Z, X) = 0$, and then calculate EZ and $\operatorname{var} Z$. Explain why Z and X are independent. (4 points)

b) Calculate the conditional distribution of Y given $X = x$ (by utilizing the result of part a, or by other means). (2 points)

Formulae

$$\Gamma(t) = \int_0^\infty x^{t-1}\, e^{-t}\, dt, \quad t > 0, \qquad \Gamma(1) = 1, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(t+1) = t\,\Gamma(t).$$

$$B(a, b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}\, dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0.$$

Pdf of $\mathrm{Gam}(\alpha, \lambda)$, where $\alpha, \lambda > 0$: $\dfrac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\lambda x}, \qquad x > 0.$

Pdf of $\mathrm{Be}(\alpha, \beta)$, where $\alpha, \beta > 0$: $\dfrac{1}{B(\alpha, \beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \qquad 0 < x < 1.$