

DEPARTMENT OF MATHEMATICS AND STATISTICS

Differential and integral calculus I.2

Final exam

21. 3. 2005

1. Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ where $f(x) = 0$ if $x < 1$ and $f(1) = 3$. Prove that the function f is Riemann-integrable on $[0, 1]$.

2. Consider the strictly increasing function $f : [0, \pi/2] \rightarrow \mathbb{R}$ defined by $f(x) = \sin x - \cos x$ for all x . Determine the integral of the inverse function

$$\int_0^1 f^{-1}(y) dy.$$

You can use the substitution $y = f(x)$.

3. Is

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2}$$

convergent? Prove your claim!

4. Consider $f_n :]-10, 10[\rightarrow \mathbb{R}$, where for all n we have $f_n(x) = x + \frac{1}{n}x^2$. Does the sequence (f_n) converge uniformly? Prove your claim!

5. Use Taylor polynomials to find a rational number q satisfying

$$\left| q - \frac{1}{e} \right| < \frac{1}{1000}.$$

Prove your claim!

INSTITUTIONEN FÖR MATEMATIK OCH STATISTIK

Differential- och integralkalkyl I.2

Final exam

24. 5. 2005

The assignments are ordered according to theme.

1. We study the Riemann-integrability of a bounded function $f : [-1, 3] \rightarrow \mathbb{R}$, where for every $x \in [-1, 3]$ we have that $f(x) = |x - 1|$. Give an example of a partition D , for which the difference between the upper sum and the lower sum, $S_D - s_D$, is less than 2^{-10} .

2. Calculate

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} dx.$$

3. Does the series

$$\sum_{k=1}^{\infty} \frac{1}{k \ln k}$$

converge? *Hint:* Using an integral helps...

4. We consider functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, where $f_n(x) = \frac{x^3}{n}$. Does the sequence converge pointwise? Is the convergence uniform?

5. Evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\sin(\sin x) - \sin x}{x^3}.$$

by using a suitable Taylor polynomial of the function $\sin t$ at the point $t = 0$.

DEPARTMENT OF MATHEMATICS AND STATISTICS

Differentiaali- ja integraalilaskenta I.2

Final examination

15. 6. 2005

1. We examine a funktion f defined on the interval $[0, 2]$, such that $f(1) = 7$ and $f(x) = 13$ when $x \neq 1$. Give an example of a decomposition D of the interval $[0, 2]$, such that $S_D - s_D < 2^{-10}$. Motivate!

2. Consider the strictly increasing function $f : [0, \pi/2] \rightarrow \mathbb{R}$, such that $f(x) = \sin x - \cos x$ for all x . Calculate the integral

$$\int_0^1 f^{-1}(y) dy$$

of the inverse function. You may for instance use the substitution $y = f(x)$.

3. Does the series

$$\sum_{k=1}^{\infty} \left(\frac{6 + \cos^4 x}{8} \right)^k?$$

converge? Motivate thoroughly.

4. Consider the functions $f_n : [-1, 3] \rightarrow \mathbb{R}$, where $f_n(x) = \frac{x^3}{n}$. Is the sequence pointwise convergent? Is the convergence uniform?

5. Resolve the limit

$$\lim_{x \rightarrow 0} \frac{\sin(\sin x) - \sin x}{x^3}.$$

A suitable Taylor expansion of $\sin t$ at $t = 0$ is helpful here.