DEPARTMENT OF MATHEMATICS AND STATISTICS
Differential and integral calculus I.1
Final exam
7. 10. 2004

1. Find

$$\lim_{n\to\infty} (\sqrt{n+3} - \sqrt{n})$$

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Prove your assertion using the definition of a limit.

2. Find

$$\lim_{x\to 0+} (\sqrt{\frac{1}{x^2} + \frac{1}{x}} - \sqrt{\frac{1}{x^2}}).$$

Prove your assertion.

3. Suppose that a function $g : R \to R$ is bounded. Consider the fuction $f : R \to R$ defined by $f(x) = x^2\ g(x)$. Prove using the definition of a derivative that f is derivable (i.e., differentiable) at $x = 0$.

4. Assume that a function $f : R \to R$ is everywhere differentiable and that $f'(x) \leq x$ holds for all x . Prove $f(2) \leq f(1) + \frac{3}{2}$. Hint: the function $g(x) = f(x) - \frac{1}{2}x^2$ is helpful.

5. Assume that a function $f :]-2, 2[\to R$ is differentiable everywhere in $]-2, 2[$. Assume moreover that $f'(-1) = 0$ and $f'(1) = 4$. Show that there is $x_0 \in]-1, 1[$ satisfying $f'(x_0) = 3$. Hint: study $f(x) - 3x$.

DEPARTMENT OF MATHEMATICS AND STATISTICS
Analysis I
Final examination
Jan 26, 2006

1. Assume that $|x - e| < 2^{-100}$ and $|x - \pi| < 2^{-100}$. Show that $|(x + y) - (e + \pi)| < 2^{-99}$.

2. Determine
$$\lim_{n \to \infty} \frac{(3n-1)(2n-2)(n-3)}{n^3}.$$
Justify your answer! (Theorems about limits of sequences from the lectures or the lecture notes may be used.)

3. Show using the definition of a limit of a function and the definition of the derivative that $f'(1) = 2$ when $f(x) = x^2$ for all x.

4. Consider the function $f\colon]0, \frac{\pi}{2}[\to \mathbb{R}$,
$$f(x) = \sqrt{\ln(1 + \sin x)}.$$
Determine the derivative of the function $f^{-1}(t)$ at $t = \sqrt{\ln(\frac{3}{2})}$. (Note that it is not necessary to determine an expression for the inverse function. Showing that f is strictly increasing is not required.)

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Assume that $f(0) > 0$ and $\lim_{x \to \infty} f(x) = 0$ and that $\lim_{x \to -\infty} f(x) = 0$. Show that there exists an a such that $f'(a) = 0$.