DEPARTMENT OF MATHEMATICS AND STATISTICS
Calculus I.1
Mid-term examination 2
13. 12. 2004

1. Using the definition for the limit of a function, show that

$$\lim_{x \to 1} \frac{x+1}{x^2+1} = 1.$$

2. We consider the function $f\colon]0,\pi[\to \mathbb{R}$

$$f(x) = \ln \sqrt{1 + \sin^2 x}.$$

(a) Compute the derivative of f.
(b) At which point(s) x does f have a local maximum?

3. We consider the function

$$f(x) = \frac{e^{\sin x}}{e^x + e^{-x}}.$$

Show that f attains its maximum value, i.e. that there exists a point a, such that for all $x \in \mathbb{R}$ the inequality $f(x) \leq f(a)$ holds. You may use all properties of the sine,- and exponential functions which are known from school, for instance the fact that $e^y \to \infty$ when $y \to \infty$. (*N.B.*: the assignment has nothing to do with derivatives.)

4. Assume that $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are continuous, and that $f'(x)$ and $g'(x)$ exist for all $x \in]0,1[$. Furthermore, assume that $f(0) > g(0)$ and that for all $x \in]0,1[$ the inequality $f'(x) \geq g'(x)$ holds. Show that $f(1) > g(1)$.

DEPARTMENT OF MATHEMATICS AND STATISTICS
Calculus I.2
Mid-term examination 2
12. 5. 2005

1. Does the improper integral
$$\int_0^\infty \frac{x}{e^{x^2}}\,dx$$
converge? If it does, calculate its value.

2. Does the series
$$\sum_{k=1}^{\infty} \frac{(\sin x + \cos x)^2 + 1}{k}$$
converge? Motivate carefully.

3. We consider functions $f_n : [0,1] \to \mathbb{R}$, where
$$f_n(x) = \sin x + n\sin(\frac{x}{n^2}).$$
Show that the sequence (f_n) converges uniformly in the set $[0,1]$.

4. Form the Taylor polynomial $\mathrm{T}_2(x;1)$ of the function $f(x) = e^x$, and using it, calculate the limit
$$\lim_{x\to 1} \frac{e^x - ex}{(x-1)^2}.$$
Motivate carefully.