Matematiikan ja tilastotieteen laitos
Algebraic Topology
Department of Mathematics and Statistics
May 13, 2008

1. Prove that if the chain maps $f, g\colon K \to L$ are chain homotopic, then

$$f_* = g_*\colon H_n(K) \to H_n(L), \quad \text{for every} \quad n \in \mathbb{Z}.$$

2. Let (X, A) and (Y, B) be topological pairs, and suppose that $f\colon (X, A) \to (Y, B)$ is a continuous map, such that $f\colon X \to Y$ and $f|A\colon A \to B$ are homotopy equivalences. Prove that

$$f_*\colon H_n(X, A) \to H_n(Y, B)$$

is an isomorphism, for each $n \geq 0$.

3. State and prove Browner's fixed–point theorem.

4. State and prove the Fundamental Theorem of Algebra.

5. Let $B \subset S^n$ be a subset of S^n, $n \geq 1$, and supppose that B is homeomorphic to S^k, where $0 \leq k \leq n-1$. Prove that the reduced homology of $S^n - B$ is given by

$$\tilde{H}_j(S^n - B) \cong \begin{cases} 0, & \text{if } j \neq n-k-1, \\ \mathbb{Z}, & \text{if } j = n-k-1. \end{cases}$$

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Algebraic Topology
Loppukoe, 14.8.2008
Matemtiikan ja tilastotieteen laitos

1. Prove the following form of the five-lemma. Let

$$\begin{tikzcd} C_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & C_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & C_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & C_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & C_5 \arrow[d, "f_5"] \\ D_1 \arrow[r, "\beta_1"] & D_2 \arrow[r, "\beta_2"] & D_3 \arrow[r, "\beta_3"] & D_4 \arrow[r, "\beta_4"] & D_5 \end{tikzcd}$$

be a commutative diagram of abelian groups and homomorphisms, where the horizontal rows are exact sequences, and assume that f_1, f_2, f_4 and f_5 are isomorphisms. Prove that f_3 is an isomorphism.

2. Prove that the inclusion $i\colon (E_+^{n-1}, S^{n-2}) \to (S^{n-1}, E_-^{n-1})$ induces isomorphisms

$$i_*\colon H_k(E_+^{n-1}, S^{n-2}) \to H_k(S^{n-1}, E_-^{n-1}),$$

for all $k \geq 0$, and $n \geq 1$. Here, for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $\|x\| = \sqrt{x_1^2, + \dots + x_n^2}$, and then $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, $E_+^{n-1} = \{x \in S^{n-1} \mid x_n \geq 0\}$, $E_-^{n-1} = \{x \in S^{n-1} \mid x_n \leq 0\}$, and $S^{n-2} = \{x \in S^{n-1} \mid x_n = 0\}$.

3. Prove that the euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

4. Show that if $f\colon S^n \to S^n$, $n \geq 1$, is a map without fixed points, then $\deg(f) = (-1)^{n+1}$.

5. Prove the following result. Let $B \subset S^n$ be a subset of S^n, $n \geq 0$, such that B is homeomorphic to S^k, where $0 \leq k \leq n-1$. Then

$$\tilde{H}_q(S^n - B) \cong \begin{cases} 0, & q \neq n-k-1 \\ \mathbb{Z}, & q = n-k-1. \end{cases}$$

Algebraic Topology I

Loppukoe 12.11.2008

Matematiikan ja tilastotieteen laitos

1. Let $0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0$ be a short exact sequence of chain complexes. Show that the sequence

$$\cdots \xrightarrow{\Delta} H_n(K) \xrightarrow{\alpha_*} H_n(L) \xrightarrow{\beta_*} H_n(M) \xrightarrow{\Delta} H_{n-1}(K) \longrightarrow \cdots, \; n \geq 1,$$

is exact at $H_n(L)$.

2. Suppose that $f : (X, A) \longrightarrow (Y, B)$ is a continuous map of topological pairs, such that $f : X \longrightarrow Y$ and $f|A : A \longrightarrow B$ are homotopy equivalences. Prove that

$$f_* : H_n(X, A) \longrightarrow H_n(Y, B)$$

are isomorphisms, for all $n \geq 0$.

3. Prove that the euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

4. Formulate and prove Brouwer's fixed-point theorem.

5. Give a proof of the following fact. Suppose $f: S^n \longrightarrow S^n$, $n \geq 1$, is a continuous map with no fixed points, then $\deg(f) = (-1)^{n+1}$.