Matematiikan laitos
Algebrallinen topologia I
Loppukoe
20.1.2004

1. Let X be a non–empty path connected space. Show that $H_0(X) \cong \mathbb{Z}$.

2. Let $f: X \to Y$ be a continuous map between topological spaces. Show that f induces a chain map $f_\#$, and that $f_\#$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$, for every $n \geq 0$.

3. Let $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ be a short exact sequence of chain complexes. Prove that the sequence

$$\cdots \to H_{n+1}(M) \xrightarrow{\Delta} H_n(K) \xrightarrow{\alpha_\#} H_n(L) \xrightarrow{\beta_\#} H_n(M) \xrightarrow{\Delta} H_{n-1}(K) \to \cdots$$

is exact at $H_n(L)$.

4. Prove that the euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

5. Let $f: S^n \to S^n$, $n \geq 1$, be a map without fixed points. Prove that $\deg(f) = (-1)^{n+1}$.

Matematiikan laitos
Algebrallinen topologia I
Loppukoe
16.3.2004

1. a) Let X be a topological space. Describe the construction of the n:th singular homology group $H_n(X)$, $n \geq 0$.
b) Determine the homology groups $H_n(\{p\})$, $n \geq 0$.

2. Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

be a commutative diagram of abelian groups and homomorphisms, where the horizontal sequences are exact. Assume that f_1, f_2, f_4 and f_5 are isomorphisms. Prove that f_3 is an isomorphism.

3. Let $f: (X, A) \to (Y, B)$ be a continuous map such that both $f: X \to Y$ and $f|A: A \to B$ are homotopy equivalences. Show that

$$f_*: H_n(X, A) \to H_n((Y, B)$$

are isomorphisms, for all $n \geq 0$.

4. Let A be a subset of a topological space X, and suppose that A is a retract of X. Show that $H_n(A)$ is a direct summand of $H_n(X)$, $n \geq 0$. That is, show that there exists an isomorphism

$$H_n(A) \oplus G \xrightarrow{\cong} H_n(X), \quad n \geq 0,$$

were G denotes some abelian group.

5. Prove that there does not exist a continuous map $f: S^{2n} \to S^{2n}$ such that $f(x)$ is orthogonal to x for all $x \in S^{2n}$, $n \geq 1$.

Matematiikan ja tilatotieteen laitos
Algebrallinen topologia II
Loppukoe
18.5.2004

1. Prove the Fundamental Theorem of Algebra.

2. Give a precise statement of the theorem concerning the existence of the long exact Mayer–Vietoris sequence. Describe the main steps in the proof of this theorem.

3. State and prove the Jordan–Brouwer separation theorem.

4. Let V be an open subset of S^n, $n \geq 1$, and let $f: V \to S^n$ be a continuous map. Let $y \in S^n$ be such that $f^{-1}(y)$ is compact.
   a) Give a precise definition of $\deg_y f$, i.e., of the local degree of f over y. For example, explain where the assumption that $f^{-1}(y)$ is compact is used.
   b) Prove that if $f: S^n \to S^n$, $n \geq 1$, is a continuous map, then $\deg_y f$ is defined for every $y \in S^n$, and moreover

$$\deg_y f = \deg(f), \text{ for all } y \in S^n.$$

Matematiikan ja tilastotieteen laitos
Algebrallinen topologia II
Loppukoe
18.6.2004

1. Prove the Fundamental Theorem of Algebra.

2. Give the main parts of the proof of the following result. Let X_1 and X_2 be subsets of a topological space X. Then the couple $\{X_1, X_2\}$ is excisive if and only if the inclusion $i_1: (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ induces isomorphisms

$$(i_1)_*: (X_1, X_1 \cap X_2) \to H_n(X_1 \cup X_2, X_2), \text{ for all } n \geq 0.$$

3. Prove the following result. Let $B \subset S^n$ be a subset of S^n such that B is homeomorphic to S^k, where $0 \leq k \leq n-1$. Then

$$\tilde{H}_j(S^n - B) \cong \begin{cases} 0, & j \neq n-k-1, \\ \mathbb{Z}, & j = n-k-1. \end{cases}$$

4. a) Let V be an open subset of S^n, and let $f: V \to S^n$ be a continuous map, $n \geq 1$. Let $y \in S^n$ be such that $f^{-1}(y)$ is compact. Define the local degree of f over y.
b) Let $f: V \to S^n$ be as above, and suppose that W is a connected, open subset of S^n such that $f|: f^{-1}(W) \to W$ is a proper map. Show that then $\deg_y f$ is defined for every $y \in W$, and that

$$\deg_{y'} f = \deg_{y''} f, \text{ for any } y', y'' \in W.$$

Matematiikan ja tilastotieteen laitos
Algebrallinen topologia I
Loppukoe
11.11.2004

1. Let X be a topological space.
a) Define and describe the singular chain complex $S(X)$ of X.
b) Define the n:th singular homology group $H_n(X)$, $n \geq 0$.

2. Let

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5
\end{array}$$

be a commutative diagram of abelian groups and homomorphisms, where the horizontal sequences are exact. Assume that f_1, f_2, f_4 and f_5 are isomorphisms. Prove that f_3 is an isomorphism.

3. Let $f: (X, A) \to (Y, B)$ be a continuous map such that both $f: X \to Y$ and $f|A: A \to B$ are homotopy equivalences. Show that

$$f_*: H_n(X, A) \to H_n(Y, B)$$

are isomomorphisms, for all $n \geq 0$.

4. Prove that the euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homomorphic if and only if $n = m$.

5. Prove that there does not exist a continuous map $f: S^{2n} \to S^{2n}$ such that $f(x)$ is orthogonal to x for all $x \in S^{2n}$, $n \geq 1$.

Matematiikan ja tilastotieteen laitos
Algebrallinen topologia I
Loppukoe
26.1.2006

1. Let $0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0$ be a short exact sequence of chain comples. Consider the sequence of homology groups

$$\cdots \longrightarrow H_{n+1}(L) \xrightarrow{\beta_*} H_{n+1}(M) \xrightarrow{\Delta} H_n(K) \xrightarrow{\alpha_*} H_n(L) \longrightarrow \cdots$$

where Δ denotes a connecting homomorphism, for the above short exact sequence of chain complexes. Prove that the above sequence of homology groups is exact at $H_n(K)$.

2. a) Let $f: X \to Y$ be a homotopy equivalence between two topological spaces. Prove that

$$f_*: H_n(X) \to H_n(Y)$$

is an isomorphism, for every $n \geq 0$.

b) Then prove that if $f: (X, A) \to (Y, B)$ is a map of pairs, such that $f: X \to Y$ and $f|A: A \to B$ are homotopy equivalences, then the induced homomorphisms

$$f_*: H_n(X, A) \to H_n(Y, B)$$

are isomorphisms, for every $n \geq 0$.

3. a) Show that the inclusion $i: S^{n-1} \to \mathbb{R}^n - \{0\}$, $n \geq 1$, is a homotopy equivalence.

b) Prove that two euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, $n \geq 0$, $m \geq 0$, are homeomorphic if and only if $n = m$.

4. a) Define the degree $\deg(f)$ of a map $f: S^n \to S^n$, $n \geq 1$, and establish the following properties:
(i) $\deg(\mathrm{id}) = 1$.
(ii) If $f, g: S^n \to S^n$, then $\deg(g \circ f) = \deg(g)\deg(f)$.
(iii) $\deg(c) = 0$, if $c: S^n \to S^n$ is a constant map.
(iv) If the two maps $f, g: S^n \to S^n$ are homotopic, then $\deg(f) = \deg(g)$.
(v) If $f: S^n \to S^n$ is a homotopy equivalence, then $\deg(f) = \pm 1$.

b) Let $f: S^n \to S^n$ be a map which is not surjective. Show that $\deg(f) = 0$.

Matematiikan ja tilastotieteen laitos
Algebrallinen topologia II
Loppukoe
22.5.2006

1. Prove the Fundamental Theorem of Algebra.

2. State and prove the Jordan–Brouwer separation theorem.

3. State and prove the Invariance of domain theorem. (You may choose either one of the two versions.)

4. Let V be an open subset of S^n, $n \geq 1$, and let $f\colon V \to S^n$ be a continuous map. Let $y \in S^n$ be such that $f^{-1}(y)$ is compact.
a) Give a precise definition of $\deg_y f$, i.e., of the local degree of f over y.
b) Prove that if $f\colon V \to S^n$, is injective, then $\deg_y f = \pm 1$, for every $y \in f(V)$.
c) Prove that if $f\colon S^n \to S^n$, $n \geq 1$, is a continuous map, then $\deg_y f$ is defined for every $\in S^n$, and moreover $\deg_y f = \deg(f)$, for all $y \in S^n$.