Algebra II

Final exam 13.5.2008

Answer four of the the five questions below. (A field means always a commutative field.)

1. Let G be a finite group that acts on a set E, $a \in E$, G_a be the stabilizer of a and Ga be the orbit of a. Show, that $Card(G) = Card(G_a)Card(Ga)$, where $Card(X)$ means the number of elements of X.

2. Determine all groups G of order 6 (up to isomorphism). (Hint: First, show that if x, y and xy are all of order 2, then x and y commute and by considering $< x, y >$ deduce that G has an element a of order 3. Now keeping in mind that if $(G : H) = 2$ then H is normal, observe that if $x \in G- < a >$ then $b = x^3 = xx^2 \notin < a >$ and b is of order 2. Finally, using conjugation, determine the possibilities.)

3. Show, that the subring $A = \{x + yi | \, x, y \in \mathbf{Q}\}$ of the field of complex numbers is isomorfic to the field of fractions of the subring $B = \{x + yi | \, x, y \in \mathbf{Z}\}$.

4. Assume that $K \subseteq E$ are fields and $a \in E$ is algebraic over K. Show that $K[a]$ is a field. (Hint: R/I is a field if I is a maximal ideal of a commutative ring R.)

5. Assume that E is a splitting field of $X^3 - 2 \in \mathbf{Q}[X]$. Find a basis for E (as a module over $\mathbf{Q}$) and show that $Gal(E/\mathbf{Q})$ is isomorphic to the symmetric group of a three element set.