HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

Department of Mathematics
and Statistics
Algebra I
Final exam, 14.8.2008

1. Show that the relation E defined on the set $\mathbb{R}$ of the real numbers by

$$xEy \iff y = ax \text{ for some } a > 0,$$

is an equivalence relation. How many different equivalence classes does this relation have?

2. The law of composition $\top$ on the set A has a neutral element, and

$$a\top(b\top c) = (a\top c)\top b \qquad \text{for all } a, b, c \in A.$$

Show that $\top$ is associative and commutative.

3. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the map defined by the formula $f(m, n) = 3m - 2n$. Show that f is a surjective homomorphism of additive groups, whose kernel $\mathrm{Ker}(f)$ is the cyclic subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $(2, 3)$.

4. Show that $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$.

5. Find all the irreducible elements of degree at most 3 of the polynomial ring $\mathbb{Z}_2[x]$ (there should be 5 of them). Is the polynomial $x^4 + x^2 + 1 \in \mathbb{Z}_2[x]$ of degree 4 irreducible?

Department of Mathematics and Statistics
Algebra I
Final exam 12.5.2009

1. Think the integers to be defined as *formal* (that is, as written, not yet as "executable") differences $m - n$ of natural numbers with the identity condition

$$m - n = p - q \iff m + q = n + p.$$

a) Show that this relation of integers expressed with the symbol $=$ is really (only) an equivalence relation in the set of formal differences of natural numbers.

b) By which formulas are the addition and multiplication of integers now defined with the aid of the addition and multiplication of natural numbers? (Think on the basis of the composition rules of $\mathbb{Z}$ how the formulas should be.)

c) Show that this addition of integers is well-defined: if the differences to be added are replaced by identical differences, then the resulting sum is identical with the original sum.

2. Let G be a (multiplicative) group and H, K and L finite subgroups of G with orders (order = the number of elements) $|H| = 15$, $|K| = 21$ and $|L| = 35$. Show that $H \cap K \cap L = \{1_G\}$.

3. Let $G = S_{\mathbb{N}}$ be the permutation group of the set $\mathbb{N}$ of natural numbers, that is, the set of the bijections $f\colon \mathbb{N} \to \mathbb{N}$ with the composition of maps as the group composition. Let us call a bijection $f \in G$ *almost the identity* if for some number $m_f \in \mathbb{N}$ it holds that $f(n) = n$ for all $n \geq m_f$. Let H be the set of the bijections $f \in G$ which are almost the identity. Show that H is a normal subgroup of G.

4. Let R be a ring, and let $Y_2(R)$ be the set of all 2-row upper triangular matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ $(a, b, c \in R,\ 0 = 0_R)$ in R equipped with the standard matrix addition and multiplication:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ 0 & c + c' \end{pmatrix}; \qquad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}.$$

Let us keep as known that then $Y_2(R)$ is a ring (this can be proved just as in the case of matrices in $\mathbb{R}$; there really is no need for an assumption that R should be commutative). Show now that the set $J = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\} \subset Y_2(R)$ is an ideal of the ring $Y_2(R)$ and that the factor ring $Y_2(R)/J$ is isomorphic with the product ring $R \times R$; give also such an isomorphism. **Quidance.** Apply the ring homomorphism theorem to the map $f\colon Y_2(R) \to R \times R$ for which $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$.

5. Find an expression for the polynomial $f = x^4 - 5x^3 + x^2 - 2x - 15$ as a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.