University of Helsinki
Department of Mathematics and Statistics

Algebra I
Final examination
20. 4. 2004

1. Give an example of a group generated by two elements which is not cyclic.

2. Solve the number congruence

$$x^2 - 3x + 2 \equiv 0 \pmod{21}.$$

3. Let $(G, \cdot)$ be a group and $\sim$ a congruence, i.e., for every $x, x', y, y' \in G$ we have that $x \sim x'$ and $y \sim y'$ imply $xy \sim x'y'$. Prove that the equivalence class of the neutral element forms a normal subgroup.

4. Determine if the mapping $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 4x^3 - 3x$, is a ring homomorphism from the ring $(\mathbb{Z}, +, \cdot)$ to itself.

5. Prove that $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ is a field where $\mathbb{Q}(\sqrt{2}) = \{\, x + y\sqrt{2} \mid x, y \in \mathbb{Q} \,\}$.

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Algebra I
Final examination
10. 8. 2004

1. Below you can see the multiplication table of the six-element group $(G, \cdot)$. Find the subgroups of this group. Does $(G, \cdot)$ have a subgroup with four elements?

$\cdot$	e	r	s	a	b	c
e	e	r	s	a	b	c
r	r	s	e	b	c	a
s	s	e	r	c	a	b
a	a	c	b	e	s	r
b	b	a	c	r	e	s
c	c	b	a	s	r	e

2. Solve the congruence equation

$$x^2 - 2x + 2 \equiv 0 \pmod{37}.$$

3. Determine if the groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{R}, +)$ isomorphic? (Here, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.)

4. a) Define the notions of a ring homomorphism and its kernel.
b) Give an example of a ring homomorphism with a non-trivial kernel.

5. Let $(K, +, \cdot)$ be a field. Suppose that for all $a, b \in K$, we have

$$(a + b)^3 = a^3 + b^3.$$

Prove that the characteristic of the field $(K, +, \cdot)$ is 3, i.e. $x + x + x = 0$ for all $x \in K$.

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Algebra I
Final examination
10.8.2004

1. Define a binary operation $*$ on $\mathbb{R}$ by

$$a * b = a^3 - 7b,$$

for $a, b \in \mathbb{R}$. Is $(\mathbb{R}, *)$ a group?

2. Find the subgroups of the symmetric group $(\mathscr{S}(\{0,1,2\}), \circ)$ on the set $\{0,1,2\}$ (or, in short, $\mathscr{S}_3$).

3. Give an example of a group $(G, *)$ and a homomorphism h from $(G, *)$ to itself such that h is not an isomorphism.

4. A king promises half of his kingdom against the following task: For each of the 64 squares of the chessboard, the employee should work a certain amount of whole days. For the first square, the work is exactly one day. For the other squares, the number of working days is always the double of the number for the previous square. Supposing the work starts at Monday and continues without breaks even in the weekend, what it the weekday of the last working day?

5. a) Let $(K, +, \cdot)$ be a field. Represent the equation related to division by a polynomial by a polynomial with coefficients in K obtained by the Division Algorithm. The condition for the remainder should be represented, whereas the proof of the result may be omitted.
b) Do the polynomial equations

$$x^{121} - 2x^{120} + 3x^{100} + 5x^{21} - 10x^{20} + 33 = 0$$

and

$$x^{21} - 2x^{20} + 3 = 0$$

have common roots in the set of reals $\mathbb{R}$?

Algebra 1

Answer the following questions:

October 25, 2005

1. Prove that if a is relatively prime to b but a divides bc, then a divides c. (Recall that two integers are said to be "relatively prime" if the only positive integer which divides them is 1.) Hint: Recall that if a and b are relatively prime, then there are integers m and n such that $ma + nb = 1$.

2. Find a cyclic subgroup of the symmetric group S_4.

3. Prove that the only ideals of a field are the two trivial ones.

4. Prove that there is only one group of order 15. Hint: use the Sylow theorems: 1. The number of p-Sylow subgroups of a group divides the order of the group, and 2. The number of p-Sylow subgroups of a group G is of the form $1 + kp$.

5. Prove that if R is a commutative ring with unit element and M is an ideal of R, then M is maximal if and only if R/M is a field.

Department of Mathematics and Statistics
Algebra 1
Final Exam
25.1.2007

(1) **Subgroups.**
  (a) Give the definition of a subgroup.
  (b) Give one subgroup criterion (you do not need to prove it).
  (c) Let G be a group and let H and K be two subgroups of G. Show that the intersection $H \cap K$ is a subgroup of G. (6p.)

(2) **Group homomorphisms.**
  (a) Let G an G' be two groups. Give the definition of a group homomorphism $f\colon G \to G'$.
  (b) Let G be a finite group and assume that $f\colon G \to \mathbb{Z}$ is a group homomorphism. Show that $\ker(f) = G$.
  (c) Let G be an abelian group and $f\colon G \to G'$ be a group homomorphism. Show that the group $\mathrm{im}(f)$ is abelian. (6p.)

(3) **Cyclic groups.**
  (a) What does the classification theorem about cyclic groups say?
  (b) What are the generators of group $\mathbb{Z}_{25}$?
  (c) Find *all* cyclic subgroups of $\mathbb{Z}_{25}$.
  (d) Are there any other subgroups of $\mathbb{Z}_{25}$? If yes, which? If no, then why not?
  (e) Draw the latice diagram of $\mathbb{Z}_{25}$. (8p.)

(4) **Lagranges theorem.**
  (a) What does Lagrange's Theorem say?
  (b) Assume that G is a finite group with $|G| = p$ where p is a prime number. Show that G has only trivial subgroups.
  (c) In the above case, show that G is a cyclic group. (6p.)

(5) **Left cosets and normal subgroups.**
  (a) What is the definition of a left coset?
  (b) What is the definition of a normal subgroup?
  (c) Let $f\colon G \to G'$ be a group homomorphism. Show that $\ker(f)$ is a normal subgroup of G. (6p.)

Department of Mathematics and Statistics
Algebra I
Final examination January 25, 2007

1. In the set $G = \{(x, y) \in \mathbf{R}^2 \mid x \neq 0\}$ a binary operation $*$ is defined by

$$(x, y) * (x', y') = (xx', yx' + y').$$

Show that $(G, *)$ is a group.

2. What is the order of the group $(\mathbf{Z}_{12}, +)$? Compute the orders of all the elements of that group.

3. (a) Let $(G, \circ)$ be an arbitrary group. Solve the equation

$$a \circ x \circ b \circ c \circ x = a \circ b \circ x.$$

(b) Apply the result of (a) to the special case $G = S_3$, when

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

4. Show that a ring R is commutative if and only if

$$(x + y)^2 = x^2 + y^2 + 2xy \text{ for all } x, y \in R.$$

5. Show that the rings $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbf{Z}\}$ and $\mathbf{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbf{Z}\}$ are not isomorphic. (*Hint:* Suppose that $f : \mathbf{Z}[\sqrt{2}] \to \mathbf{Z}[\sqrt{5}]$ is a ring homomorphism. This means that (1) $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbf{Z}[\sqrt{2}]$ and (2) $f(1) = 1$. Consider $f(\sqrt{2})$. The fact that $\sqrt{5}$ is irrational can be used without proof.)