An Introduction to Microlocal Analysis with Applications to Inverse Problems

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Lecture 1

Microlocal analysis is used inverse problems to detect singularities. Microlocal analysis is local analysis in phase space.

References:

- A. Grigis, J. Sjöstrand: Microlocal Analysis for Differential Operators
- PDE notes

Example 1. *X-ray tomography and Radon-transform. Measure attenuation of X-rays as they go through Ω. Beer's law: Intensity $I(x)$ of X-ray at x (along line)*

$$\frac{dI}{dx} = -\sigma I(x).$$

Denote I_R = intensity at receiver and I_S = intensity at source. Then

$$I_R(x) = I_S(x) e^{-\int_L \sigma(x)\,dx}.$$

If we measure I_R and I_S we can recover $\int_L \sigma\,dx$. Inverse problem: Can we recover $\sigma(x), x \in \Sigma$ by measuring $\int_L \sigma(x)\,dx$ for all lines L? 1917 Radon considered this problem.

Let $\omega \in S^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}$ and $H_{s,\omega} = \{x \in \mathbb{R}^n \colon x \cdot \omega = s\}$, case $n = 2$ is lines. Here ω = normal to hyperplane and s = distance to origin.

$C_0^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \colon \operatorname{supp} f \text{ compact}\}$.

Define

$$Rf(s,\omega) = \int_{x\cdot\omega = s} f(x)\,dH,$$

$f \in C_0^\infty(\mathbb{R}^n)$ and dH = standard Lebesgue measure on $H_{s,\omega}$.

Problem 1. *Show that $dH ds = dx$.*

Note that $Rf(-s,-\omega) = Rf(s,\omega)$ for all $s \in \mathbb{R}$ and $\omega \in S^{n-1}$.

Problem 2. *Show that $R \colon C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ is linear and continuous.*

Denote by R^t the formal transpose of R:

$$\begin{aligned}(Rf, g) &= \int_{S^{n-1}} \int_{\mathbb{R}} Rf(s,\omega) g(s,\omega)\,ds d\omega = \int_{\mathbb{R}^n} f(x) R^t g(x)\,dx \\ &= (f, R^t g).\end{aligned}$$

($dw =$ standard Lebesgue measure on S^{n-1}) We have

$$\int_{S^{n-1}} \int_{\mathbb{R}} Rf(s,\omega) g(s,\omega)\, ds d\omega = \int_{S^{n-1}} \int_{\mathbb{R}} \int_{x\cdot\omega = s} f(x)\, dH g(s,\omega)\, ds d\omega$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(x) \underbrace{\left(\int_{S^{n-1}} g(x\cdot\omega,\omega)\, d\omega \right)}_{=R^t g} dx.$$

Definition 2. *Let $g \in C_0^\infty(\mathbb{R} \times S^{n-1})$. We define*

$$R^t g(x) = \int_{S^{n-1}} g(x\cdot\omega,\omega)\, d\omega.$$

R^t is also called backprojection operator.

Inverse problem: Can we find f if we know $Rf(s,\omega) \forall s \in \mathbb{R}, \omega \in S^{n-1}$? Operator $R^t R$ is called normal operator.

Problem 3. *Show that*

$$R^t Rf(x) = c_n \int \frac{f(y)}{|x-y|} dy = c_n f * \frac{1}{|x|},$$

where $*$ is convolution.

Note $\widehat{R^t Rf}(\xi) = c_n \hat{\frac{1}{|x|}} \hat{f}$.

Problem 4. *Show that $\widehat{R^t Rf}(\xi) = \frac{c_n}{|\xi|^{n-1}} \hat{f}(\xi) = \mathscr{F}(c_n(-\Delta)^{-(n-1)/2} f)(\xi)$.*

Recall

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\dot{\xi}} |\xi|^\alpha \hat{f}(\xi) d\xi,$$

for $-n < \alpha, f \in C_0^\infty(\mathbb{R}^n)$. We get

Theorem 3 (Radon inversion formula, RIF)**.**

$$f = c_n (-\Delta)^{(n-1)/2} R^t Rf,$$

for $f \in C_0^\infty(\mathbb{R}^n)$.

Operator $(-\Delta)^{-(n-1)/2} = \sqrt{-\Delta}$ is not local,

$$\sqrt{-\Delta} f(x) = \int f(y) |x-y|\, dy.$$

Remark 4. *For $n = 2$ Radon inversion formula is not local. To find $f(x_0)$ need to know $\int_L f$ for all lines.*

Remark 5. *Radon inversion formula is extended to $f \in \mathscr{E}'(\mathbb{R}^n)$ = distributions with compact support:*

$$(-\Delta)^{(n-1)/2} R^t R f = c_n f$$

for all $f \in \mathscr{E}'(\mathbb{R}^n)$.

Case $n = 2$:

$$\begin{aligned}
&(-\Delta)^{1/2} R^t R f = c_n f \\
&\Rightarrow (-\Delta)^{-1/2} (-\Delta)^{1/2} R^t R f = (-\Delta)^{-1/2} f \\
&\Rightarrow (-\Delta) R^t R f = c_n (-\Delta)^{-1/2} f
\end{aligned}$$

Local inversion formula, recover $(-\Delta)^{-1/2} f$.

"Singularities" of $(-\Delta)^{-1/2} f$ = "singularities" of f.

Extend R, R^t to distributions [1].

$$(Rg, g) = \langle f, R^t g \rangle,$$

$f \in C_0^\infty(\mathbb{R}^n), g \in C_0^\infty(\mathbb{R} \times S^{n-1})$.

Let $X \subseteq \mathbb{R}^n$, open. $\mathscr{D}(X)$ = distributions on $X = \{u \colon C_0^\infty(\mathbb{R}^n) \to \mathbb{C}, \text{linear, continuous}\}$ and $\operatorname{supp} u$ = complement of largest open set where $u = 0$.

Let $x_0 \in X$.

$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0),$$

δ_{x_0} = Dirac delta at x_0. Then $\operatorname{supp} \delta_{x_0} = \{0\}$.

Let $u \in \mathscr{E}'(X)$. We define

$$\langle Ru, \varphi \rangle = \langle u, R^t f \rangle,$$

$\varphi \in C_0^\infty(\mathbb{R} \times S^{n-1})$.

Note: $\varphi \in C_0^\infty(\mathbb{R} \times S^{n-1})$, $R^t \varphi \in C^\infty(\mathbb{R}^n)$ but $R^t \varphi$ may not be compactly supported.

Problem 5. *$n = 2$: Find $\varphi \in C_0^\infty \mathbb{R} \times S^1$ such that $R^t \varphi$ is not compactly supported.*

[1] Reference: e.g. G. Friedlander: Introduction to the Theory of Distributions

Let $v \in \mathscr{D}'(\mathbb{R} \times S^{n-1}) = \{v \colon C_0^\infty(\mathbb{R} \times S^{n-1}, \text{linear, continuous}\}$. We define

$$\langle R^t v, \varphi \rangle = \langle v, R\varphi \rangle$$

$\varphi \in C_0^\infty(\mathbb{R}^n)$.

Problem 6. *Show that*

$$u = c_n(-\Delta)^{(n-1)/2} R^t R u = u$$

for all $u \in \mathscr{E}'(\mathbb{R}^n)$.

Extend RIF to more general curves.

$(-\Delta)^{1/2}$ is a prototype of a pseudodifferential operator which is *elliptic*. (Note that differential operators are local, this is not) Inverse of $(-\Delta)^{1/2}$ is $(-\Delta)^{-1/2}$ (Fourier side: $|\xi|\hat{f} = 1/|\xi|\hat{f}$).

Singularities of distributions

Let $u \in \mathscr{D}'(X), X \subseteq \mathbb{R}^n$, X open.

Definition 6.

$$\begin{aligned} \operatorname{singsupp} u &= \textit{singular support of } u \\ &= \textit{complement of largest open set where } u \textit{ is smooth} \end{aligned}$$

$u \in C^\infty(V), V \subseteq X$, *u smooth in V if*

$$\langle u, \varphi \rangle = \int_X f(x)\varphi(x)\, dx,$$

$\varphi \in C_0^\infty(V), f \in C^\infty(V)$.

Example 7. *a)* $x_0 \in X$, $\operatorname{singsupp} \delta_{x_0} = \{x_0\}$.

b) Heavyside function

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

$\operatorname{singsupp} H = \{0\}$.

c) for $f \in C^k(X), f \notin C^\infty(X)$ singsupp *may by very large.*

Let $u \in \mathscr{D}'(X)$.

$$\operatorname{WF} u = \text{Wave front set of } u \subseteq X \times (\mathbb{R}^n - \{0\}).$$

Proposition 8. *Suppose $u \in \mathscr{E}'(X)$. Then*

$$u \text{ is smooth} \Leftrightarrow |\hat{u}(\xi)| = O(|\xi|^{-N}) \forall N \in \mathbb{N}.$$

Formally

$$u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi.$$

Look at directions where $\hat{u}$ is not rapicly decaying.

Definition 9 (Wave front set)**.** *Suppose $X \subseteq \mathbb{R}^n$, X open, $u \in \mathscr{D}'(X)$.*

$$\begin{aligned}(x_0, \xi_0) \notin \mathrm{WF}\, u \Leftrightarrow & \exists U \textit{ neighborhood of } x_0 \\ & \exists V \textit{ neighborhood of } \xi_0 \\ & \textit{s.t.} \quad \widehat{\varphi u}(t\xi) = O(t^{-N}) \forall N \in \mathbb{N}, \forall \varphi \in C_0^\infty(U), \forall \xi \in V\end{aligned}$$

Example 10. *a) $u = \delta_{x_0}$, $x_0 = 0$, $\delta_0 \in \mathscr{D}'(\mathbb{R}^n)$. Claim:* $\mathrm{WF}(\delta_0) = \{(0, \xi) \colon \xi \in \mathbb{R}^n - \{0\}\}$

Proof:

$$\widehat{\varphi\delta_0}(\xi) = \langle \varphi\delta_0, e^{-i\langle \cdot, \xi\rangle} \rangle = \varphi(0),$$

$\varphi \in C_0^\infty(\mathbb{R}^n)$, can be non-zero.

b) $\mathbb{R}^2, x = (x_1, x_2)$.

$$\langle \delta_{x_2=0}, \varphi \rangle = \int \varphi(x_1, 0)\, dx,$$

$\varphi \in C_0^\infty(\mathbb{R}^n)$. Claim: $\mathrm{WF}(\delta_{x_2=0} = \{(x1, 0), (0, \xi_2), \xi_2 \neq 0\}$.

Proof: For $\xi = (\xi_1, \xi_2) \neq 0$

$$\begin{aligned}\widehat{\varphi\delta_{x_2=0}}(t\xi) &= \langle \varphi\delta_{x_2=0}, e^{-i\langle \cdot, t\xi\rangle} \rangle = \langle \delta_{x_2=0}, \varphi e^{-i\langle \cdot, t\xi\rangle} \rangle \\ &= \int e^{-ix_1 t\xi_1} \varphi(x_1, 0)\, dx_1.\end{aligned}$$

$\xi_1 \neq 0$:

$$\widehat{\varphi\delta_{x_2=0}}(t\xi) = O(t^{-N}) \forall N \in \mathbb{N}.$$

Fourier transform of φ w.r.t. x_1 and φ is smooth, hence the decay.

$\xi_1 = 0$:

$$\widehat{\varphi\delta_{x_2=0}}(t\xi) = \int \varphi(x_1, 0) x_1$$

Lecture 2

About the topology of $C_0^\infty(X)$ and $C^\infty(X)$, $X \subseteq \mathbb{R}^n$ open.

Definition 11. *$f_j \in C_0^\infty(X), f_j \to 0$ if*

1. *$\exists K \subseteq X$, K compact such that* $\operatorname{supp} f_j \subseteq K$
2. *$\partial^\alpha f_j \to 0$ uniformly in K $\forall\alpha$*

Seminorms: Suppose $\cup_{j=1}^\infty K_j = X$, K_j compact, $K_j \subseteq K_{j+1}$. Define

$$p_j, k(f) = \sup_{x\in K_j, |\alpha|\le k} |\partial^\alpha f(x)|$$

Seminorms generate topology of $C^\infty(X)$.

Problem 7. *In $\mathbb{R}^2$ let*

$$u = \chi_{B(0,1)} \begin{cases} 1, & |x| < 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Show that

$$\operatorname{WF} u = \{(x, \xi) \colon x \in \partial B(0,1), x \cdot \xi = 0\}.$$

Problem 8. *In $\mathbb{R}^2$ let $u = \chi_{|x_1|\le 1, |x_2|\le 1}$. Find* $\operatorname{WF} u$.

Proposition 12. $\operatorname{WF} u$ *is closed and conic.*

Conic: $(x, \xi) \in \operatorname{WF} u \Rightarrow (x, \lambda\xi) \in \operatorname{WF} u, \lambda > 0$.

Problem 9. *Let $F \subseteq X \times (\mathbb{R}^n - \{0\})$ be a close and conic set. Find $u \in D'(X)$ such that* $\operatorname{WF} u = F$.

Proposition 13. *Let $\pi \colon X \times (\mathbb{R}^n - \{0\}) \to X$, $(x, \xi) \mapsto x$. Then* $\pi(\operatorname{WF} u) = \operatorname{singsupp} u$.

Wave front set is natural generalisation of singular support.

Sketch of proof. Based on $u \in C^\infty(x)$, $\widehat{\varphi u}(t\xi) O(t^{-N}) \forall \xi \in \mathbb{R}^n - \{0\}, \forall N \in \mathbb{N}, \varphi \in C_0^\infty(X)$ □

Pseudodifferential operators

$(-\Delta)^{1/2}$ is the prototype of a pseudodifferential operator,

$$(-\Delta)^{1/2} = \int e^{ix\cdot\xi}|\xi|\hat{f}(\xi)\,d\xi,$$

$f \in C_0^\infty(\mathbb{R}^n)$.

Definition 14. *Symbols.* $a \in S^m(X \times \mathbb{R}^n) \subset C^\infty(X \times \mathbb{R}^n), m \in \mathbb{N}$ *if* $\forall K \Subset X$ *(*$= K$ *compact)* $\forall \alpha, \beta$ *multi-index* $\exists C_{\alpha,\beta,K} > 0$ *such that*

$$\sup_{x\in K} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\beta|}$$

Note: w.r.t. variable ξ mimics properties of homogeneous functions.

Example 15. $a \in C^\infty(X \times \mathbb{R}^n)$ *such that* $a(x,\lambda\xi) = \lambda^m a(x,\lambda\xi)$, $\lambda > 0$. *Then* $\partial_{\xi_i} a(x,\lambda\xi) = \lambda^{m-1}a(x,\xi)$ *and* $\partial_\xi^\beta a(x,\lambda\xi) = \lambda^{m-|\beta|}a(x,\xi)$.

Note that $|\xi|$ is not a symbol in this sense. We can write

$$(-\Delta)^{1/2} f(x) = \int e^{ix\cdot\xi}\chi(\xi)\hat{f}(\xi)\,d\xi + \int e^{ix\cdot\xi}(1-\chi(\xi))\hat{f}(\xi)\,d\xi$$

where

$$\chi(\xi) = \begin{cases} 0, & |\xi| < 1, \\ 1, & |\xi| > 2, \end{cases}$$

$\chi \in C^\infty(\mathbb{R}^n)$. Then $\chi(\xi)|\xi| \in S^1(X \times \mathbb{R}^n)$.

Function $1-\chi$ is compactly supported, so by Fubini's theorem

$$\int e^{ix\cdot\xi}(1-\chi(\xi))\hat{f}(\xi)\,d\xi = \int e^{i(x-y)\cdot\xi}(1-\chi(\xi))f(y)\,dy\,d\xi = \int K(x,y)f(y)\,dy$$

where

$$K(x,y) = \int e^{i(x-y)\xi}(1-\chi(\xi))|\xi|\,d\xi$$

is integral operator, smoothing $K\colon \mathscr{E}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Kernel $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 16. *Suppose* $K\colon C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$,

$$Kf(x) = \int \tilde{K}(x,y)f(y)\,dy,$$

$\tilde{K} \in C^\infty(X \times X)$. *Then exists* $K\colon \mathscr{E}'(X) \to C^\infty(X)$ *linear, continuous extension.*

Proof. Suppose $u \in \mathscr{E}'(X)$. Define

$$\langle Ku, \varphi \rangle = \langle u, K^t \varphi \rangle,$$

$\varphi \in C_0^\infty(X)$ and

$$K^t \varphi(x) = \int \tilde{K}(y, x) \varphi(y) \, dy.$$

Then

$$\langle u, K^t \varphi \rangle = \int \langle u, \tilde{K}(y, \cdot) \rangle \varphi(y) \, dy.$$

□

Problem 10. *Show that*

$$\langle u, K^t \varphi \rangle = \int \langle u, \tilde{K}(y, \cdot) \rangle \varphi(y) \, dy.$$

Problem 11. *Show that* $f(y) = \langle u, \tilde{K}(y, \cdot) \rangle$ *belongs to* $C^\infty(X)$.

Notation:

$$Kf(x) = \int K(x.y) f(y) \, dy$$

$K \in C^\infty(X \times X)$, is called smoothing. In particular if K is smoothing

$$\mathrm{WF}(Ku) = \emptyset$$

for all $u \in \mathscr{E}'(X)$. We will neglect smoothing operators.

Definition 17. $P \in \Psi^m(X)$, *pseudodifferential operator of order* m, *if*

$$Pf(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) \, d\xi \text{ modulo smoothing}$$

$\forall f \in C_0^\infty(X)$, *where* $p \in S^m(X \times \mathbb{R}^n)$.

Example 18. *a)* $(-\Delta)^{\alpha/2} \in \Psi^m(\mathbb{R}^n), -n < \alpha$.

b) Let $P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha f, a_\alpha \in C^\infty(X)$ *where* $D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n} f$ *and* $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$ *(so we have* $D_{x_j} e^{ix \cdot \xi} = \xi_j e^{ix \cdot \xi}$*).*

$$\begin{aligned} P(x, D)f &= \frac{1}{(2\pi)^n} \sum_{|\alpha| \le m} a_\alpha(x) \int e^{ix \cdot \xi} \xi^\alpha \hat{f}(\xi) \, d\xi \\ &= \int e^{ix \cdot \xi} \underbrace{\Big(\frac{1}{(2\pi)^n} \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha \Big)}_{\in S^m(X \times \mathbb{R}^n)} \hat{f}(\xi) \, d\xi. \end{aligned}$$

c) R = Radon transform:

$$R^t R = (-\Delta)^{-1/2} \in \Psi^{-1}(\mathbb{R}^n)$$

(up to a constant?).

Suppose $Pf(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha f(x), f \in C_0^\infty(X)$. Principal symbol of P is

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Full symbol

$$\tilde{\sigma}_m(P)(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha.$$

Definition 19. *Suppose $P \in \Psi^m(X)$,*

$$Pf(x) = \int e^{ix\cdot\xi} p(x,\xi)\widehat{f}(\xi)\,\xi \text{ modulo smoothing,}$$

$f \in C_0^\infty(X), p \in S^m(X \times \mathbb{R}^n)$. Principal symbol

$$\sigma_m(P)(x,\xi) = p(x,\xi) \text{ modulo } S^{m-1}(X \times \mathbb{R}^n)$$

Full symbol

$$\tilde{\sigma}_m(P)(x,\xi) = p(x,\xi) \text{ modulo } S^{-\infty}(X \times \mathbb{R}^n)$$

where

$$S^{-\infty}(X \times \mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(X \times \mathbb{R}^n).$$

(i.e. $\sigma_m \in S^m/S^{m-1}$ and $\tilde{\sigma}_m \in S^m/S^{-\infty}$)

Remark 20. *If $P \in \Psi^m(X)$ for all $m \in \mathbb{Z}$,*

$$Pf(x) \int e^{ix\cdot\xi} p(x,\xi)\widehat{f}(\xi)\,\xi$$

and

$$p \in \bigcap_{m\in\mathbb{Z}} S^m(X \times \mathbb{R}^n) = S^(X \times \mathbb{R}^n)$$

then P is smoothing.

$$\begin{aligned} Pf(x) &= \int e^{i(x-y)\cdot\xi} p(x,\xi) f(y)\,dyd\xi \\ &= \int K(x,y) f(y)\,dy \end{aligned}$$

where second step if by Fubini and

$$K(x,y) = \int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi \in C^\infty(X \times X).$$

Operator side	Symbol side
Ψ^m	$a(x,\xi)$
P	$\sigma_m(P)(x,\xi)$
	$\tilde{\sigma}_m(P)(x,\xi)$
P^t	?
P^*	?
PQ	?
P^{-1}	?

Definition 21. *$P \in \Psi^m(X)$ is elliptic if*

$$\sigma_m(P)(x,\xi) \neq 0 \forall (x,\xi) \in X \times (\mathbb{R}^n - \{0\}).$$

Example 22. *Operators $(-\Delta)^{\alpha/2}, -n < \alpha$ and $R^t R$ are elliptic.*

Theorem 23. *Let $P \in \Psi^m(X)$ elliptic. Then $\exists A \in \Psi^{-m}(X)$ such that*

$$AP = I + K$$
$$PA = I + \tilde{K}$$

where $K, \tilde{K}$ are smoothing.

Lecture 3

Schwartz kernel: $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^n$ open sets. $A \colon C_0^\infty(Y) \to \mathscr{D}'(X)$ linear continuous. Then $\exists! k_A \in D'(X \times y)$ such that

$$\langle A\varphi, \psi \rangle = k_A(\psi \otimes \varphi)$$

$\varphi \in C_0^\infty(Y), \psi \in C_0^\infty(X)$ and $(\psi \otimes \varphi)(x, y) = \psi(x)\varphi(y)$. k_A = Schwartz kernel.

Formally

$$A\varphi(x) = \int k_A(x, y)\varphi(y)\, dy, \varphi \in C_0^\infty(Y).$$

Example 24. *A = Identity, $X = Y$.*

$$A\varphi(x) = \varphi(x) = \int k_A(x, y)\varphi(y)\, dy,$$

$k_A = \delta(x - y)$.

Problem 12. *Suppose $A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, a_\alpha \in C_0^\infty(X)$. Show that*

$$k_A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \delta(x - y).$$

What about $A \in \Psi^m$, k_A = ? Formally

$$Pf(x) = \int e^{ix\cdot\xi} p(x, \xi) \widehat{f}(\xi)\, \xi + K \text{ smoothing}$$

and

$$\int e^{ix\cdot\xi} p(x, \xi) \widehat{f}(\xi)\, \xi = \int e^{i(x-y)\cdot\xi} p(x, \xi) f(y)\, dy d\xi$$

Note that p might not be integrable Fubini could not be used.

Formally

$$k_P(x, y) = \int e^{i(x-y)\cdot\xi} p(x, \xi)\, d\xi.$$

So called oscillatory integral, $k_P \in \mathscr{D}'(X \times X)$?

Note that

$$(I - \Delta_y)^M e^{i(x-y)\cdot\xi} = (1 + |\xi|^2)^M e^{i(x-y)\cdot\xi}$$

Using this and integration by parts M times

$$\begin{aligned}\langle k_P, \varphi\rangle &= \int e^{i(x-y)\cdot\xi} p(x,\xi)\varphi(x,y)\,dxdyd\xi \\ &= \int e^{i(x-y)\cdot\xi} \frac{p(x,\xi)}{(1+|\xi|^2)^M}(I-\Delta_y)^M\varphi(x,y)\,dxdyd\xi,\end{aligned}$$

$\varphi \in C_0^\infty(X\times X)$. We have

$$\frac{p(x,\xi)}{(1+|\xi|^2)^M} \in S^{m-2M}(X\times\mathbb{R}^n)$$

For M sufficiently large, $m-2M<-n$, makes sense.

Definition 25.

$$\langle k_P, \varphi\rangle = \int e^{i(x-y)\cdot\xi} \frac{p(x,\xi)}{(1+|\xi|^2)^M}(I-\Delta_y)^M\varphi(x,y)\,dxdyd\xi,$$

$M>(n+m)/2$.

Problem 13. *Show that definition is independent of M as long as $M>(m+n)/2$.*

Problem 14. *Let $\eta\in C_0^\infty(\mathbb{R}^n), \eta(\xi)=1, |\xi|\le 1$. Show that*

$$\lim_{\varepsilon\to 0+}\int \eta(\varepsilon\xi)e^{i(x-y)\cdot\xi}p(x,\xi)\,dx = k_P$$

as defined.

Note K smoothing, $k_K\in C^\infty(X\times X)$.
Suppose $A\in\Psi^m(X)$, $A^t=?$.
$k_{A^t}(x,y)=k_A(y,x)$

$$k_A(x,y)\int e^{i(x-y)\cdot\xi}p(x,\xi)\,\xi$$

(ignoring smoothing part, for that we know)

$$\begin{aligned}k_{A^t}(x,y) &= \int e^{i(y-x)\cdot\xi}p(y,\xi)\,d\xi \\ &= \int e^{i(x-y)\cdot(-\xi)}p(y,\xi)\,d\xi \\ &\overset{\xi\to-\xi}{=} \int e^{i(x-y)\cdot\xi}p(y,-\xi)\,d\xi\end{aligned}$$

We want

$$\int e^{i(x-y)\cdot\xi} p(y,-\xi) f(y)\, d\xi dy = \int e^{ix\cdot\xi} p^t(x,\xi)\hat{f}(\xi)\, d\xi + \text{smoothing},$$

$f \in C_0^\infty(X), p^t \in S^m(X\times\mathbb{R}^n)$.

If we had

$$\int e^{i(x-y)\cdot\xi} p(x,-\xi) f(y)\, dy d\xi = \int e^{ix\cdot\xi} \underbrace{p(x,-\xi)}_{\in S^m(X\times\mathbb{R}^n)} \hat{f}(\xi)\, d\xi.$$

and we write $p(y,-\xi) = p(x,-\xi) + (y-x)\underbrace{h(x,y,\xi)}_{S^m(X\times X\times\mathbb{R}^n)}$ then

$$\begin{aligned}
&\int e^{i(x-y)\cdot\xi} p(y,-\xi) f(y)\, dy d\xi \\
&= \int e^{i(x-y)\cdot\xi} p(x,-\xi) f(y)\, dy d\xi + \int e^{i(x-y)\cdot\xi} (y-x) h(x,y,\xi) f(y)\, dy d\xi \\
&= \int e^{ix\cdot\xi} p(x,-\xi)\hat{f}(y)\, d\xi + \int (-D_\xi) e^{i(x-y)\cdot\xi} h(x,y,\xi) f(y)\, dy d\xi \\
&= \int e^{ix\cdot\xi} p(x,-\xi)\hat{f}(y)\, d\xi + \int e^{i(x-y)\cdot\xi} \underbrace{D_\xi h(x,y,\xi)}_{\in S^{m-1}(X\times X\times\mathbb{R}^n)} f(y)\, dy d\xi.
\end{aligned}$$

We develop in Taylor series $D_\xi h(x,y,\xi)$ around $x=y$

$$D_\xi h(x,y,\xi) = (x-y)\underbrace{\tilde{h}(x,y,\xi)}_{\in S^{m-1}(X\times X\times\mathbb{R}^n)}.$$

Inductively

$$\begin{aligned}
&\int e^{i(x-y)\cdot\xi} p(y,-\xi) f(y)\, dy d\xi = \int e^{ix\cdot\xi} p(x,-\xi)\hat{f}(\xi)\, d\xi \\
&+ \int e^{ix\cdot\xi} \partial_y D_\xi p(x,-\xi)\hat{f}(\xi)\, d\xi + \cdots + \int e^{ix\cdot\xi} \underbrace{\frac{\partial_x^M D_\xi^M p(x,-\xi)}{M!}}_{\in S^{m-M}(X\times\mathbb{R}^n)} \hat{f}(\xi)\, d\xi \\
&+ \int e^{i(x-y)\cdot\xi} h_M(x,y,\xi) f(y)\, dy d\xi,
\end{aligned}$$

$h_M \in S^{m-M}(X\times X\times\mathbb{R}^n)$.

Definition 26. $p_j \in S^{m-j}(X \times \mathbb{R}^n)$, $j = 0, 1, \dots, M, \dots$. $\exists p \in S^m(X \times \mathbb{R}^n)$ *such that*

$$p \sim \sum_{j=0}^{\infty} p_j$$

in the sense that

$$p - \sum_{j=0}^{M} p_j \in S^{m-M}(X \times \mathbb{R}^n).$$

Lemma 27. *Given* $p_j \in S^{m-j}(X \times \mathbb{R}^n)$, $j = 0, 1, \dots, M, \dots$ $\exists! p \in S^m(X \times \mathbb{R}^n)$ *modulo* $S^{-\infty}(X \times \mathbb{R}^n)$ *such that*

$$p \sim \sum_{j=0}^{\infty} p_j.$$

Note similar to $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $f \sim \sum_{n=0}^{\infty} a_n x^n$.

Theorem 28. *Let* $P \in \Psi^m(X)$,

$$Pf(x) = \int e^{ix\cdot\xi} p(x,\xi) \widehat{f}(\xi)\, \xi + K\, smoothing,$$

$p \in S^m(X \times \mathbb{R}^n)$. *Then* $P^t \in \Psi^m(X)$

$$\begin{aligned} \sigma_m(P^t)(x,\xi) &= \sigma_m(P)(x,-\xi), \\ \tilde{\sigma}_m(P^t)(x,\xi) &\sim \sum_{M=0}^{\infty} \frac{\partial_x^M D_\xi^M p(x,-\xi)}{M!}. \end{aligned}$$

Adjoint $A^* \in \Psi^M(X)$?

$$\begin{aligned} Kf(x) &= \int K(x,y) f(y)\, dy, \\ K^* f(x) &= \int \overline{k(y,x)} f(y)\, dy. \end{aligned}$$

$A \in \Psi^m(X)$.

$$\begin{aligned} k_A(x,y) &= \int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi, \\ k_{A^*}(x,y) &= \int e^{i(x-y)\cdot\xi} \overline{p}(x,\xi)\, d\xi. \end{aligned}$$

Theorem 29. *Let $A \in \Psi^m(X)$, then $A^* \in \Psi^m(X)$ and*

$$\sigma_m(A^*)(x,\xi) = \overline{\sigma_m(A)(x,\xi)},$$
$$\tilde{\sigma}_m(A^*)(x,\xi) \sim \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \partial_x^\alpha D_\xi^\alpha \overline{p(x,\xi)}.$$

Composition of Pseudodifferential operators

$A \in \Psi^m(X), B \in \Psi^{\tilde{m}}(X) \Rightarrow AB \in \Psi^{m+\tilde{m}}(X)$?

Suppose A and B are smoothing

$$k_{AB}(x,z) = \int \underbrace{k_A(x,y)}_{\in C^\infty} \underbrace{k_B(y,z)}_{\in C^\infty} \, dy$$

if this makes sense.

Problem 15. *Find an example.*

We need a-priori condition on A or B.

Definition 30. *Suppose $A \in \Psi^m(X)$. Then A is called properly supported if $\forall K \Subset X$ support of $k_A(\cdot, y), y \in K$, is compact or support of $k_A(x, \cdot), x \in K$, is compact.*

Example 31. *a) $Pf(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha f(x), a_\alpha \in C^\infty(X)$. P is properly supported,*

$$k_P(x,y) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \delta(x-y).$$

b)$P \in \Psi^m(X)$,

$$k_P(x,y) = \int e^{i(x-y)\cdot\xi} p(x,\xi) \, d\xi,$$

$p \in S^m(X \times \mathbb{R}^n)$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ near 0, cut-off function.

$$k_{\tilde{P}} = \int e^{i(x-y)\cdot\xi} \chi(x-y) p(x,\xi) \, d\xi,$$

$\tilde{P} \in \Psi^m(X)$, $\tilde{P}$ is properly supported.

Lecture 4

$S \in \Psi^m(X)$.

Proposition 32. *$A = \tilde{A} + R$, where $\tilde{A} \in \Psi^m(X)$ is properly supported and R smoothing.*

Proof.

$$k_A(x,y) = \int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi,$$

$p \in S^m(X \times \mathbb{R}^n)$. Define

$$k_{\tilde{A}}(x,y) = \int e^{i(x-y)\cdot\xi} \chi(x-y) p(x,\xi)\, d\xi$$

where $\xi \in C_0^\infty(X)$, $\chi(x) = 1$ for $|x| \le 1$.

$$k_A = k_{\tilde{A}} + \int \underbrace{e^{i(x-y)\cdot\xi}(1-\chi(x-y))p(x,\xi)}_{\in C^\infty(X\times X)}\, d\xi.$$

$\operatorname{supp}(1-\chi(x-y)) \cap \{(x,x) \colon x \in X\} = \emptyset$. Using

$$D_\xi^M e^{i(x-y)\cdot\xi} = (x-y)^M e^{i(x-y)\cdot\xi}$$

and integration by parts

$$\begin{aligned}\int e^{i(x-y)\cdot\xi}(1-\chi(x-y))p(x,\xi)\, \xi &= \int \frac{D_\xi^M e^{i(x-y)\cdot\xi}}{(x-y)^M}(1-\chi(x-y))p(x,\xi)\, d\xi \\ &= (-1)^M \int \frac{e^{i(x-y)\cdot\xi}}{(x-y)^M}(1-\chi(x-y))D_\xi^M p(x,\xi)\, d\xi.\end{aligned}$$

□

Problem 16. *a) Check that*

$$\int \frac{e^{i(x-y)\cdot\xi}}{(x-y)^M}(1-\chi(x-y))D_\xi^M p(x,\xi)\, d\xi \in C^k(X \times X)$$

for any $k \in \mathbb{N}$ if M is sufficiently large.

b)

$$k_{\tilde{A}}(x,y) = \int e^{i(x-y)\cdot\xi} \chi(x-y) p(x,\xi)\, d\xi,$$

$\tilde{A} \in \Psi^m(X)$.

Proposition 33. *$A \in \Psi^m(X)$ properly supported. Then $A\colon C_0^\infty(X) \to C_0^\infty(X)$ linear, continuous.*

Theorem 34. *Suppose $A \in \Psi^m(X), B \in \Psi^{\tilde{m}}(X)$ and B is properly supported. Then $AB \in \Psi^{m+\tilde{m}}(X)$ and*

$$\sigma_{m+\tilde{m}}(AB) = \sigma_m(A)\sigma_{\tilde{m}}(B)$$
$$\tilde{\sigma}_{m+\tilde{m}} \sim \sum_\alpha \frac{1}{\alpha!} \underbrace{\partial_\xi^\alpha \tilde{\sigma}_A(x,\xi) D_x^\alpha \tilde{\sigma}_B(x,\xi)}_{\in S^{m+\tilde{m}-|\alpha|}(X\times\mathbb{R}^n)}$$

Sketch of a proof.

$$ABf = \int e^{ix\cdot\xi}\tilde{\sigma}_A(x,\xi)\widehat{BF}(\xi)\,d\xi \text{ modulo smoothing}$$

$$Bg(x) = \int e^{ix\cdot\xi}\tilde{\sigma}_B(x,\xi)\widehat{g}(\xi)\,\xi$$

Trick: $AB = A(B^t)^t$. More details in notes of PDE course. □

Definition 35. *a) $A \in \Psi^m(X)$. Let $u \in \mathscr{E}'(X)$,*

$$\langle Au, \varphi\rangle = \langle u, A^t\varphi\rangle, \varphi \in C_0^\infty(X)$$

b) $A \in \Psi^m(X)$ properly supported. Let $u \in \mathscr{D}'(X)$,

$$\langle Au, \varphi\rangle = \langle u, A^t\varphi\rangle, \varphi \in C_0^\infty(X)$$

(Note A^t is properly supported)

Problem 17. *a) $A \in \Psi^m(X), A\colon \mathscr{E}'(X) \to D'(X)$ linear, continuous.*
b)$A \in \Psi^m(X)$ properly supported, $A\colon \mathscr{D}'(X) \to D'(X)$ linear, continuous.

Theorem 36 (Pseudo-local property)**.** *$A\Psi^m(X)$. Then*

$$\mathrm{WF}(Au) \subseteq \mathrm{WF}(u) \quad \forall u \in \mathscr{E}'(X).$$

Corollary 37.

$$\mathrm{singsupp}(Au) \subseteq \mathrm{singsupp}\, u \quad \forall u \in E'(X).$$

Proposition 38.

$$\mathrm{WF}(-\Delta)^{1/2}u = \mathrm{WF}\, u \quad \forall u \in \mathscr{E}'(X).$$

Proof. By earlier theorem

$$\mathrm{WF}(-\Delta)^{1/2}u \subseteq \mathrm{WF}\,u \quad \forall u \in \mathscr{E}'(X).$$

$$(-\Delta)^{-1/2}(-\Delta)^{1/2}u = u$$

so

$$\mathrm{WF}\,u = \mathrm{WF}(-\Delta)^{-1/2}(-\Delta)^{1/2}u \subseteq \mathrm{WF}(-\Delta)^{-1/2}u$$

□

Sketch of proof of pseudo-local property.

$$k_A = \int e^{i(x-y)\cdot\xi} p(x,\xi)\,d\xi,$$

$p \in S^m(X \times \mathbb{R}^n)$. Claim: $\operatorname{singsupp} k_A(x,y) \subseteq \{(x,x) \colon x \in X\}$

$$Af(x) = \int k(x,y) f(y)\,dy$$

$$\langle k_A, \varphi \rangle = \int e^{i(x-y)\cdot\xi} p(x,\xi)\varphi(x,y)\,d\xi dx dy,$$

$\varphi \in C_0^\infty(X \times X)$. $\varphi = 0$ near $\{(x,x) \colon x \in X\}$. Use

$$\frac{D_\xi^M e^{i(x-y)\cdot\xi}}{(x-y)^M} = e^{i(x-y)\cdot\xi}.$$

Get

$$\begin{aligned}\langle k_A, \varphi \rangle &= (-1)^M \int e^{i(x-y)\cdot\xi} \underbrace{D_\xi^M(p,\xi)}_{\in S^{m-M}(X\times\mathbb{R}^n)} \varphi(x,y)\,dx dy d\xi \\ &= (-1)^M \int R_M(x,y)\varphi(x,y)\,dx dy\end{aligned}$$

$$R_M(x,y) = \int e^{i(x-y)\cdot\xi} D_\xi^M p(x,\xi)\,d\xi$$

Given any $k \in \mathbb{N}$, $R_m \in C^k(X \times X)$ for M sufficiently large.

$A \in \Psi^m(X), \operatorname{singsupp} k_A \subseteq \{(x,x) \colon x \in X\}$

$$Af(x) = \int k_A(x,y) f(y)\,dy$$

$\operatorname{singsupp} Af \subseteq \operatorname{singsupp} f$. □

Problem 18. *Show that for $A \in \Psi^m(X)$*

$$\mathrm{WF}(Af) \subset \mathrm{WF}(f) \quad \forall f \in \mathscr{E}'(X).$$

Theorem 39. *Suppose $A \in \Psi^m(X)$, elliptic. $\exists B \in \Psi^m(X)$ such that*

$$BA = I + K$$
$$AB = I + \tilde{K},$$

$K, \tilde{K}$ is smoothing. B is called parametrix.

Sketch of a proof.

$$\sigma_{-m}(B) = \frac{\chi}{\sigma_m(A)},$$

$\chi \in C_0^\infty(\mathbb{R}^N), \chi(\xi) = 1$, for $|\xi| > 2, \chi(\chi) = 0$, for $|\xi| \leq 1$.

By calculus of pseudodifferential operators (remember $(1-\chi)$ cutting gives smoothing operator)

$$\int e^{ix\cdot\xi}\chi(\xi)p(x,\xi)\,d\xi + \underbrace{\int e^{ix\cdot\xi}(1-\chi(\xi))p(x,\xi)\,d\xi}_{\text{smoothing}}$$

$$BA = I + R_{-1}$$

$R_{-1} \in \Psi^{-1}(X)$. ($\sigma_m(A) = 0 \Rightarrow A \in \Psi^{m-1}(X)$) Neumann series

$$(I + R_{-1})^{-1} = I - R_{-1} + \underbrace{R_{-1}^2}_{\in\Psi^{-2}(X)} + \cdots + (-1)^N \underbrace{R_{-1}^N}_{\in\Psi^{-N}(X)} + \cdots \in \Psi^0(X)$$

by Borel's lemma. Thus $(I + R^{-1})BA = I + K$, K smoothing. □

Corollary 40. *$A \in \Psi^m(X)$ elliptic.*

$$\mathrm{WF}(Au) = \mathrm{WF}(u) \quad \forall u \in \mathscr{E}'(X).$$

Since $BAu = u + K$, $\mathrm{WF}\,u = \mathrm{WF}(BAu) \subseteq \mathrm{WF}(Au)$.

Application

Generalized Radon transforms. Let $\omega \in S^{n-1}, a \in C^\infty(X \times S^{n-1})$.

$$R_a f(s,\omega) = \int \delta(s - \phi(x,\omega))a(x,\omega)f(x)\,dx,$$

$f \in C_0^\infty(X), \varphi \in \phi(X \times S^{-1})$, $|a| \geq c > 0$. a is a weight.

$\phi(x,\omega) = x \cdot \xi, a \equiv 1$ is the usual Radon transform.

Radon transpose: remember

$$R^t g = \int_{S^{n-1}} g(x, x \cdot \omega)\, d\omega.$$

Define

$$R_b^t g(x) = \int_{S^{n-1}} g(\phi(x,\omega),\omega) b(x,\omega)\, d\omega,$$

$b \in C^\infty(\mathbb{R}^n \times S^{n-1}), |b| \geq c > 0$ on $\mathbb{R}^n \times S^{n-1}$ (weight).

Goal: to show that under some conditions on $\{\phi(x,\omega)\}_{\omega \in S^{n-1}}$ $R_b^t R_a$ is an elliptic pseudodifferential operator, $R_b^t R_a \in \Psi^{-(n-1)/2}(\mathbb{R}^n)$. Then $\exists B \in \Psi^{(n-1)/2}(\mathbb{R}^n)$ such that $BR_b^t R_a = I +$ smoothing.

We have

$$\begin{aligned} R_a f(s,\omega) &= \int \delta(s - \phi(x,\omega)) a(x,\omega) f(x)\, dx \\ &= \int e^{i(s-\phi(x,\omega)\rho} a(x,\omega) f(x)\, dx d\rho, \end{aligned}$$

can be intepreted as oscillatory integral.

$$\begin{aligned} R_b^t R_a f(x) &= \int_{S^{n-1}} R_a f(\phi(x,\omega),\omega) b(x,\omega)\, d\omega \\ &= \int e^{i(\phi(x,\omega)-\phi(y,\omega))\rho} a(x,\omega) b(y,\omega)\, dy d\omega d\rho. \end{aligned}$$

Want to $\omega \to -\omega$ and $\rho \to -\rho$, so assume

$$\begin{aligned} a(x,-\omega) &= a(x,\omega), \\ b(x,-\omega) &= b(x,\omega), \\ \phi(x,-\omega) &= -\phi(x,\omega), \end{aligned}$$

$$R_b^t R_a f(x) = 2 \int_0^\infty \int_{S^{n-1}} e^{i(\phi(x,\omega)-\phi(y,\omega))\rho} a(x,\omega) b(y,\omega)\, d\omega d\rho f(y)\, dy$$

Polar coordinates $\xi = \omega\rho, \omega = \xi/|\xi|$,

$$R_b^t R_a f(x) = 2 \iint e^{i(\phi(x,\xi)-\phi(y,\xi))} \frac{1}{|\xi|^{n-1}} a(y, \frac{\xi}{|\xi|}) b(x, \frac{\xi}{|\xi|}) f(y)\, dy d\xi,$$

assumed/defined $\phi(x, \lambda\omega) = \lambda\phi(x, \omega), \lambda > 0$.

Taylor series at $x = y$

$$\phi(x, \xi) - \phi(y, \xi) = (x - y) \cdot h(x, y, \xi),$$

$h(x, y, \xi) = \nabla\phi(x, \xi)$. $h(x, y, \xi)$ homogeneous of degree 1 in ξ. Write $\eta = h(x, y, \xi)$.

$$R_b^t R_a f(x) = 2 \iint e^{i(x-y)\cdot\eta} \frac{1}{|h^{-1}\eta|^{n-1}} a(x, \frac{h^{-1}\eta}{|h^{-1}\eta|}) b(x, \frac{h^{-1}\eta}{|h^{-1}\eta|}) \frac{|d\xi|}{|d\eta|} \, d\eta f(y) \, dy,$$

need also condition for h to be able to change variables.

Theorem 41. *Suppose $\phi \in C^\infty(\mathbb{R}^n \times S^{n-1})$ such that $\phi(x, -\omega) = \phi(x, \omega)$, $\nabla_x \phi(x, \omega) \neq 0 \forall \omega \in S^{n-1}$ and $\left(\frac{\partial^2\phi}{\partial_{x_i}\partial_{\omega_i}}\right)$ non-singular. Suppose $a(x, \omega) \in C^\infty(\mathbb{R}^n \times S^{n-1}), |a| \geq c > 0, a(x, -\omega) = a(x, \omega)$ and same for b.*

Then $R_b^t R_a \in \Psi^{-(n-1)/2}(\mathbb{R}^n)$, elliptic.