An Introduction to Microlocal Analysis

Richard B. Melrose

Gunther Uhlmann

Massachusetts Institute of Technology

University of Washington

Contents

0.1. Introduction

One of the origins of scattering theory is the study of quantum mechanical systems, generally involving potentials. The simplest 'free space' system reduces to the study of the spectral theory for the flat Laplace operator on Euclidean space. This is briefly recalled here and in particular the relevance of the plane wave solutions, to the corresponding flat wave equation, is illustrated. In the later chapters of this book, the scattering theory for perturbations of the flat Laplacian is discussed with the initial approach being via the solution of the Cauchy problem for the corresponding perturbed wave equation. An outline of the material can be found towards the end of this Introduction.

On Euclidean space, $\mathbb{R}^n$, let $x_1, \dots, x_n$ be the standard coordinates. We shall denote by $\mathcal{C}^\infty(\mathbb{R}^n)$ the space of all infinitely differentiable functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$. The Laplacian is a differential operator which gives a linear map $\Delta : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ where

$$\Delta u(x) = \sum_{1\le j\le n} -\frac{\partial^2 u}{\partial x_j^2} = \sum_{1\le j\le n} D_j^2 = |D|^2. \tag{0.1}$$

Here we have used the notation which is convenient in dealing with the Fourier transform:

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}. \tag{0.2}$$

The fact that Δ has constant coefficients, i.e. is a polynomial in the operators D_j, allows its invertibility properties to be investigated directly using the Fourier transform. In place of $\mathcal{C}^\infty(\mathbb{R}^n)$ consider the subspace of Schwartz functions

$$\mathcal{S}(\mathbb{R}^n) = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}^n); \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty \ \forall\ \alpha, \beta \in \mathbb{N}^n \right\}. \tag{0.3}$$

Here we have used multiindex notation, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n}$. The Fourier transform

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx \tag{0.4}$$

defines an isomorphism from $\mathcal{S}$ to itself, with the inverse transform being

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(x)dx. \tag{0.5}$$

Directly from (0.4) and (0.5) the action of the Laplacian on $\mathcal{S}$ can be written

$$\Delta u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^2\hat{u}(\xi)d\xi. \tag{0.6}$$

This represents Δu as a union of the plane waves $\exp(ix\cdot\xi)$ which are eigenfunctions in the sense that

$$\Delta e^{ix\cdot\xi} = |\xi|^2 e^{ix\cdot\xi} \tag{0.7}$$

although they are never elements of the space $\mathcal{S}$. If $|\xi| = \lambda\omega$ with $\omega \in \mathbb{S}^{n-1}$, so $|\lambda|$ is the square root of the eigenvalue, then these plane waves corresponding to the eigenvalue λ^2 are the $\exp(i\lambda x \cdot \omega)$.

For $0 \neq \lambda \in \mathbb{R}$ these plane waves span, as ω varies in $\mathbb{S}^{n-1}$, the space of tempered eigenfunctions. Certainly averaging against a smooth function $h \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ gives an eigenfunction

$$u(x) = \int_{\mathbb{S}^{n-1}} e^{i\lambda x\cdot\omega} h(\omega) d\omega \tag{0.8}$$

which is bounded. In fact the principle of stationary phase allows its precise asymptotic behaviour, as $|x| \to \infty$, to be described. For $x \neq 0$, set $x = |x|\theta$ with $\theta \in \mathbb{S}^{n-1}$. Introducing these polar coordinates into (0.8) gives

$$u(|x|\theta) = \int_{\mathbb{S}^{n-1}} e^{i\lambda |x|\theta\cdot\omega} h(\omega) d\omega. \tag{0.9}$$

As a function of $\omega \in \mathbb{S}^{n-1}$ the function $\theta\cdot\omega$ is a (or really the prototypical) Morse function. It has non-vanishing gradient except at the two critical points $\omega = \pm\theta$ at which it takes its maximum and minimum values, 1 and -1 respectively. Moreover these critical points are non-degenerate with the Hessian matrix being positive-definite at the minumum and negative-definite at the maximum. Thus

$$\begin{aligned} u(r\theta) \sim e^{i\lambda r}(\frac{\lambda}{r})^{\frac12(n-1)} e^{-\frac14\pi(n-1)i}\pi^{\frac12(n+1)} \sum_{j\geq 0} r^{-j} h_j^+(\theta) \\ + e^{-i\lambda r}(\frac{\lambda}{r})^{\frac12(n-1)} e^{\frac14\pi(n-1)i}\pi^{\frac12(n+1)} \sum^{j\geq 0} r^{-j} h_j^-(\theta) \end{aligned} \tag{0.10}$$

This expansion implies in particular that the u in (0.8) is of the form

$$u(x) = e^{i\lambda|x|}|x|^{-\frac12(n-1)} f(\frac{x}{|x|}) + e^{-i\lambda|x|}|x|^{-\frac12(n-1)} g(\frac{x}{|x|}) + u',\ u' \in L^2(\mathbb{R}^n) \tag{0.11}$$

where

$$g(\theta) = A_0 f(\theta) = i^{n-1} f(-\theta). \tag{0.12}$$

In fact, as discussed in Chapter 11, given $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, there is a unique solution of $(\Delta - \lambda^2)u = 0$ of the form (0.11) and then the coefficient g is given by (0.12).

The operator A_0 in (0.12) can be viewed as the absolute scattering matrix for Euclidean space; it is generally normalized away. The stationary approach to scattering theory proceeds by showing (generally using perturbation theory) that the perturbed equation, for example that obtained by adding a potential to the Laplacian, again has a unique solution of the form (0.11) for each $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ but with the replationship (0.12) altered. Rather than use this stationary approach we shall use a hyperbolic approach in which these solutions are constructed uniformly in λ, rather than separately for each λ. The advantage of this approach is that it gives directly the asymptotic behaviour of the scattering matrix for large λ.

The hyperbolic, or dynamical, approach to scattering theory is based on the fact that the plane wave eigenfunctions of the Laplacian can be obtained by inverse Fourier transformation of the distributions $\delta(t - x\cdot\omega)$:

$$e^{i\lambda x\cdot\omega} = \int_{\mathbb{R}} e^{it\lambda}\delta(t - x\cdot\omega). \tag{0.13}$$

The 'travelling' or 'progressing' wave $\delta(t - x\cdot\omega)$ is a solution of the wave equation for Euclidean space:

$$(D_t^2 - \Delta)\delta(t - x\cdot\omega) = 0. \tag{0.14}$$

Here δ is the one-dimensional Dirac δ distribution, so by definition

$$\langle \delta(t - x\cdot\omega), \phi\rangle = \int_{\mathbb{R}^n} \phi(x\cdot\omega, x)dx,\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}). \tag{0.15}$$

The first part of this book deals with the construction of the forward fundamental solution to the perturbed wave operator, $D_t^2 - \Delta - V$, and to the existence and elementary properties of the scattering kernel. Various concepts from microlocal analysis, in particular that of a conormal distribution, are interwoven into this discussion.

In Chapter 1 we start the construction of perturbed progressing waves which are to be solutions of the wave for $\Delta + V$, where V is a potential, similar to this solution, $\delta(t - x\cdot\omega)$, of the free equation. Initially we construct solutions modulo smooth errors. In Chapter 2 the Radon transform is introduced, this is used later as an effective tool in the study of solutions of the wave equation. In Chapter 3 a more general study of distribution with regularity properties analogous to those of $\delta(t - z\cdot\omega)$ is begun, this is a first step towards microlocal analysis. In Chapter 4 the approximate plain wave solutions obtained in Chapter 1 are combined to give a parametrix for the Cauchy problem for the perturbed wave operator. This involves an integration over the angular parameters and in Chapter 5 the related general operations of pull-back and push-forward on the conormal distributions of Chapter 3 are described. In Chapter 6 an iterative argument is used to obtain the forward fundamental solution of the wave operator from the parametrix constructed in Chapter 1; this finally allows the existence of the perturbed wave solutions to be shown. Again in Chapter 7 computational aspect of the general operations, analogous to those by which the forward fundamental solution is obtained from the perturbed plain waves, are considered. The wave group, Lax-Phillips transform and hence the scattering kernel are examined in Chapter 8. The latter is an example of a conormal distribution associated to a hypersurface of codimension greater than 1, i.e. not a hypersurface, and these are considered more generally in Chapter 9. In Chapter 10 the more traditional scattering amplitude is introduced and studied.

In the second part of the book various questions related to the scattering matrix and amplitude are examined. Initially, in Chapter 11 the dynamical approach to scattering theory of the first part is related to more traditional stationary scattering theory. As a consequence of the dynamical approach the high-frequency behaviour of the scattering amplitude is discussed and in particular it is shown that the scattering amplitide determines the potential perturbation. In Chapter 12 the more refined (but less constructive) result that the scattering amplitude at any fixed positve energy determines the potential is deduced. In Chapter 13 the degree to which the backscattering determines the potential is also considered. In Chapter 14 the two trace formulæ of scattering theory, relating the regularized trace of the wave group on the one hand to the determinant of the scattering matrix and on the other to the poles of the analytic continuation of the scattering matrix are derived. Estimates on the distribution of the these scattering poles can be found in Chapter 15.

In the third part of the book we turn to a discussion of the more geometric problem of scattering by a metric perturbation of Euclidean space. Much the same approach is used as for the potential perturbation treated initially, with the emphasis being on the extra geometric problems. In Chapter 16 Hamilton-Jacobi theory is introduced, to allow geodesics for the perturbed metric and bicharacteristics for the perturbed wave equation to be investigated. In Chapter 17 the fundamental problem of the parametrization of Lagrangian submanifolds is treated and used in Chapter 18 to develop the theory of Lagrangian distributions. This theory is then used to show that the wave group for the Laplacian of the perturbed metric is a Fourier integral operator, i.e. has a Lagrangian distribution as its Schwartz kernel. This in turn is used in Chapter 19 to show that the scattering kernel in this case is a Lagrangian distribution with singularities corresponding exactly to the geometric scattering of geodesics; related inverse problems are also discussed.

More on aims and philosophy at the beginning
Where are pseudodifferential operators?

Part 1

Potential scattering

CHAPTER 1

Progressing waves

If $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is the (possibly complex-valued) potential then we wish to find a solution to the continuation problem:

$$\begin{aligned} P_V u = (D_t^2 - \Delta - V)u(t,x;\omega) = 0 \text{ in } \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{S}_\omega^{n-1} \\ u(t,x;\omega) = \delta(t - x\cdot\omega) \text{ in } t < -\rho. \end{aligned} \tag{1.1}$$

Here the constant ρ is such that $\mathrm{supp}(V) \subset \{|x| \le \rho\}$ from which it follows that

$$t - x\cdot\omega < 0 \text{ if } x \in \mathrm{supp}(V) \text{ and } t < -\rho. \tag{1.2}$$

Of course if $V \equiv 0$ then (**??**) shows that $\delta(t - x\cdot\omega)$ is itself a solution to this problem.

Once we have shown the existence and uniqueness of the solution to (1.1), subject to (1.2), (this will take some time) it will be used, in Chapter 10 to define the scattering kernel as follows. From the methods we use to solve (1.1) it follows that $u(t,x;\omega)$ can itself be represented as a superposition of plane waves:

$$u(t,x;\omega) = \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} \delta(s - x\cdot\theta)\alpha_V(t,s,\theta;\omega)ds d\omega. \tag{1.3}$$

The distribution $\alpha_V(t-s,s,\theta;\omega)$ turns out to be independent of s in $s > \rho$. The most important part of α_V is therefore

$$\kappa_V(t,\theta,\omega) = \alpha_V(t-\rho,\rho,\theta;\omega) \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}); \tag{1.4}$$

κ_V is called the scattering kernel (of V). It is the main object of study below. Clearly, in case the potential vanishes identically, there is a trivial representation (1.3):

$$\begin{aligned} V \equiv 0 \Longrightarrow \\ \alpha_0(t,s,\theta;\omega) = \delta(t-s)\delta_\omega(\theta),\ \kappa_0(t,\theta,\omega) = \delta(t)\delta_\omega(\theta) \end{aligned} \tag{1.5}$$

as one would expect. Although it is not completely obvious from (1.3), the scattering kernel is determined by the $u(t,x;\omega)$ in the exterior region, $|x| > R$ for any R. In fact it can be recovered from these plane wave solutions by a limiting process at spatial infinity, in terms of Friedlander's radiation field:

$$\kappa_V(t,\theta,\omega) = \lim_{r\to\infty} r^{\frac{n-1}{2}} \partial_t u(t+r, r\theta;\omega). \tag{1.6}$$

Thus the difference, $\kappa_V(t,\theta,\omega) - \delta(t)\delta_\omega(\theta)$ represents the 'far-field' effect of the potential. This and related matters are discussed in Chapter 11.

Our first step towards solving (1.1) is to construct a solution up to smooth errors. In fact, in order to subsequently remove these smooth error terms, it is convenient to solve a more general problem, allowing initial data of plane wave type on the initial surface $t = t_0$. Since the analysis required to solve the general initial value problem is only a slight extension of that for the continuation problem we shall deal with (1.1) first.

In case $V \equiv 0$, $u = \delta(t - x \cdot \omega)$ solves (1.1) and can be thought of as a plane wave moving in the direction ω. We therefore look for a solution of (1.1) of the form:

$$u = \delta(t - x \cdot \omega) + H(t - x \cdot \omega) g(t, x, \omega), \tag{1.7}$$

where H is the Heaviside function

$$H(r) = \begin{cases} 1 & r \geq 0 \\ 0 & r < 0 \end{cases} \tag{1.8}$$

and $g \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1})$ is a smooth function. The presence of the Heaviside factor in (1.7) means that the 'wave' is only non-zero after the initial front, represented by $\delta(t - x \cdot \omega)$, has passed i.e. in $t \geq x \cdot \omega$. Of course the Heaviside term introduces singularities into u and the idea is that these additional singularities should account for the perturbation of the plane wave produced by the potential V.

Proposition 1.1. *Suppose $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, then a function $g \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1})$ can be so chosen that u, defined by* (1.7)*, satisfies*

$$P_V u = D_t^2 u - \Delta u - V u = f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1}) \tag{1.9}$$

and the continuation condition

$$\begin{aligned} u = \delta(t - x \cdot \omega) \ \text{in } t < C \ \text{where} \\ C = \inf\{x \cdot \omega; x \in \operatorname{supp}(V), \omega \in \mathbb{S}^{n-1}\}. \end{aligned} \tag{1.10}$$

Proof. The first step is to compute the form of $P_V u$, when u is defined by (1.7) for some $\mathcal{C}^\infty$ function g. Of course

$$P_V \delta(t - x \cdot \omega) = -V \delta(t - x \cdot \omega). \tag{1.11}$$

Using Leibniz' formula the additional terms are:

$$\begin{aligned} P_V H(t - x \cdot \omega) g =& (P_0 H) \cdot g \\ &+ 2\{D_t H \cdot D_t g - \sum_{k=1}^{n} D_k H \cdot D_k g\} + H \cdot P_V g, \end{aligned} \tag{1.12}$$

where $H = H(t - x \cdot \omega)$. Since $P_0 H = 0$ this can be rewritten

$$\begin{aligned} &P_V [H(t - x \cdot \omega) g(t, x, \omega)] \\ &\quad = -2i\{\omega \cdot D_x g + D_t g\} \delta(t - x \cdot \omega) + H(t - x \cdot \omega) \cdot P_V g. \end{aligned} \tag{1.13}$$

To cancel off the δ-term in (1.13) we must choose g to satisfy

$$2i\{\omega \cdot D_x g + D_t g\} = -V \text{ on } t = x \cdot \omega. \tag{1.14}$$

Once (1.14) is satisfied we have

$$P_V u = H(t - x \cdot \omega) P_V g. \tag{1.15}$$

This is a $\mathcal{C}^\infty$ function precisely when the smooth coefficient, $P_V g$, of the Heaviside function vanishes with all its derivatives on the surface $t = x \cdot \omega$. Clearly this is a condition on the Taylor series of g at this surface. Indeed if we consider the Taylor series of g :

$$\sum_{k=0}^{\infty} \frac{1}{k!}(t - x\cdot\omega)^k g_k(x,\omega), \quad g_k = \partial_t^k g(x\cdot\omega, x, \omega) \tag{1.16}$$

then the Taylor series of $P_V g$ is just

$$\begin{gathered}\sum_{k=0}^{\infty} \frac{1}{k!}(t - x\cdot\omega)^k h_k(x,\omega), \\ h_k = -2i\omega\cdot D_x g_{k+1} - (\Delta + V)g_k \ \forall\ k \geq 0.\end{gathered} \tag{1.17}$$

Thus to make $P_V u$ in (1.15) smooth we need to solve the recursive equations

$$2i\omega\cdot D_x g_k = -(\Delta + V)g_{k-1}, \text{ for } k \geq 1. \tag{1.18}$$

Notice that, in terms of the Taylor series (1.16), the first equation, (1.14), also takes the form (1.18) for $k = 0$ provided that we set $g_{-1} = 1$ i.e. (1.14) is equivalent to

$$2i\omega\cdot D_x g_0(x,\omega) = -V(x). \tag{1.19}$$

We also need to satisfy the continuation condition (1.10). This certainly implies that

$$g_k(x,\omega) = 0 \text{ in } x\cdot\omega < C,\ \forall\ k \geq 0. \tag{1.20}$$

Taken together the equations (1.18) and (1.20) have a unique solution. Since $g_{-1} = 1$ the first equation is just

$$2\omega\cdot\partial_x g_0 = -V,\ g_0 = 0 \text{ in } x\cdot\omega < C. \tag{1.21}$$

This can be integrated directly to give

$$g_0(x,\omega) = -\frac{1}{2}\int_{-\infty}^{x\cdot\omega} V(x + (r - x\cdot\omega)\omega)dr. \tag{1.22}$$

Similarly the higher order equations (1.18), for $k \geq 1$, can be integrated using the initial conditions (1.20):

$$g_k(x,\omega) = -\frac{1}{2}\int_{-\infty}^{x\cdot\omega} (\Delta + V)g_{k-1}(x + (r - x\cdot\omega)\omega, \omega)dr,\ k \geq 1. \tag{1.23}$$

With these choices of the g_k we conclude that the Taylor series in (1.17) is trivial. Now, Borel's Lemma allows us to choose $g \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^{n-1})$ with the Taylor series (1.16), with the g_k given by (1.22) and (1.23).

Lemma 1.1. *(E.Borel) If* $\{g_k\}$ *with* $g_k \in \mathcal{C}^\infty(\mathbb{R}^p)$ *is an arbitrary sequence of* $\mathcal{C}^\infty$ *functions then there exists* $g \in \mathcal{C}^\infty(\mathbb{R}^{p+1})$ *such that*

$$\partial_{p+1}^k g(x,0) = g_k(x) \quad \forall\ k \in \mathbb{N}. \tag{1.24}$$

Proof. Although this is a standard result we give a proof since 'completeness' results of this type are important later as well. We start with the simple case where

$p = 0$ so the sequence g_k is a sequence in $\mathbb{C}$ and we need to construct a $\mathcal{C}^\infty$ function $g \in \mathcal{C}^\infty(\mathbb{R})$ with

$$\frac{d^k g}{dx^k}(0) = g_k, \ k = 0, 1, \dots . \tag{1.25}$$

To construct g choose a cut-off function $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ with $0 \le \rho(x) \le 1$, $\rho(x) = 1$ if $|x| \le \frac{1}{2}$ and $\rho(x) = 0$ if $|x| > 1$. Then for any sequence $\epsilon_k > 0$ with $\epsilon_k \downarrow 0$

$$g(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} g_k \rho(x/\epsilon_k) \tag{1.26}$$

defines a $\mathcal{C}^\infty$ function in $x \neq 0$. Indeed if $x \neq 0$ only a finite number of the terms in (1.26) are non-zero. We shall show that an appropriate choice of the ϵ_k makes the series converge uniformly in $|x| \le 1$, with all its derivatives, to a solution of (1.25).

Differentiating (1.26), in $x \neq 0$, gives

$$\frac{d^p g}{dx^p} = \sum_{k>p} \sum_{j \le p} \frac{x^{k-p+j}}{(k-p+j)!} g_k \frac{\rho^{(j)}(x/\epsilon_k)}{\epsilon_k^j} + h_p(x) \tag{1.27}$$

where $h_p(x)$ is $\mathcal{C}^\infty$. Each derivative of ρ is uniformly bounded, $|\rho^{(j)}(s)| < C_p$, and $|x| \le \epsilon_k$ on the support of $\rho^{(j)}(x/\epsilon_k)$ so

$$\sum_{j \le p} \frac{|s|^{k-j}}{(k-p+j)!} \frac{|\rho^{(j)}(x/\epsilon_k)|}{\epsilon_k^j} \le C_p' \epsilon_k^{k-p}, \ C_p' = pC_p. \tag{1.28}$$

Thus the series in (1.27) converges uniformly, by comparison with a geometric series, provided

$$\epsilon_k^{k-p} |g_k| \le 2^{-k} \text{ if } k > p. \tag{1.29}$$

For each value of k, (1.29) represents only a finite number of conditions:

$$\epsilon_k \le \min_{p<k} \left(\frac{2^{-k}}{|g_k| + 1} \right)^{1/(k-p)} . \tag{1.30}$$

Since the minimum is positive for each k the sequence ϵ_k can be chosen so that (1.30) holds, and hence each of the series (1.27) converges absolutely and uniformly. This proves that (1.25) holds.

Returning to the general case of the lemma it is enough to assume that all the g_k have supports in a fixed ball, since the general construction can proceed using a partition of unity. Note from (1.30) that the ϵ_k can be chosen so that (1.27) converges as desired uniformly even when the g_k vary in a bounded set for each k (possibly depending on k). This allows the ϵ_k to be chosen so that all the x_{p+1} derivatives of

$$g(x', x_{p+1}) = \sum_{k=0}^{\infty} \frac{x_{p+1}^k}{k!} g_k(x') \rho(x_{p+1}/\epsilon_k), \ x' = (x_1, \dots, x_p) \tag{1.31}$$

converge uniformly and absolutely. The same type of argument shows that ϵ_k can be chosen so that (1.31) converges as a series in $\mathcal{C}^\infty(\mathbb{R}^{p+1})$ to a solution of (1.24). $\square$

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FIGURE 1. Plane wave initial data

Now to construct g we need to 'sum' the power series at $t = x \cdot \omega$. To do this using Lemma 1.1, introduce $s = t - x \cdot \omega$ and x, ω as independent variables and choose

$$h \in \mathcal{C}^\infty(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}) \text{ with } \partial_s^k h(0, x, \omega) = g_k(x, \omega) \ \forall\ k. \tag{1.32}$$

By construction all terms in the Taylor series vanish in $x \cdot \omega < C$ so h can be chosen to vanish there too. Then set $g(t, x, \omega) = h(t - x \cdot \omega, x, \omega)$ and consider u defined by (1.7). Clearly this is a solution of the continuation problem (1.9), (1.10). □

More generally we consider the initial value problem on the initial surface $t = t_0$. We shall drop the assumption that V has compact support and also allow it to depend on the time variable t. We wish to find a solution to

$$\begin{gathered} P_V u \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_{t_0} \times \mathbb{R}_s \times \mathbb{S}^{n-1}) \text{ with} \\ u|_{t=t_0} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1}) \text{ and} \\ D_t u|_{t=t_0} - \delta(s - x \cdot \omega) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1}). \end{gathered} \tag{1.33}$$

Thus the variables t_0, s and ω are parameters as far as the differential equation is concerned. Even in the case $V \equiv 0, s = 0$ there is no solution to (1.33) corresponding to a single progressing wave as in (1.7), rather we need to consider two plane waves passing through $s = x \cdot \omega$ at $t = t_0$. Thus we look for a solution in the form

$$\begin{aligned} u(t, x, t_0, s, \omega) = H(t - t_0 + s - x \cdot \omega) &g_+(t, x, t_0, s, \omega) \\ + H(t - t_0 - s + x \cdot \omega) &g_-(t, x, t_0, s, \omega) \\ \text{with } g_\pm \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1}). \end{aligned} \tag{1.34}$$

The coefficients $g_\pm$ are to be chosen so that (1.33) holds. For the differential equation we can use (1.13) on each term:

$$\begin{aligned} P_V u = -2i\{\omega \cdot D_x g_+ + D_t g_+\}&\delta(t - t_0 + s - x \cdot \omega) \\ - 2i\{\omega \cdot D_x g_- + D_t g_-\}&\delta(t - t_0 - s + x \cdot \omega) \\ + H(t - t_0 + s - x \cdot \omega)&P_V g_+ \\ + H(t - t_0 - s + x \cdot \omega)&P_V g_-. \end{aligned} \tag{1.35}$$

Thus the conditions on $g_\pm$ imposed by the differential equation in (1.33) are just conditions on the Taylor series at $t = t_0 \mp (s - x \cdot \omega)$ respectively:

$$\begin{gathered} \pm\omega \cdot D_x g_{\pm,0} = 0 \\ \pm\omega \cdot D_x g_{\pm,k} = \\ \frac{i}{2}\left[\Delta g_{\pm,k-1} + \sum_{\ell \le k-1} \binom{k-1}{\ell} V_{\pm,k-1-\ell}\, g_{\pm,\ell}\right] \end{gathered} \tag{1.36}$$

where

$$\begin{aligned} g_{\pm,k}(x, t_0, s, \omega) &= \partial_t^k g_\pm(t_0 \mp (s - x \cdot \omega), x, t_0, s, \omega),\ k = 0, 1, \ldots \\ V_{\pm,p}(x, t_0, s, \omega) &= \partial_t^p V(t_0 \mp (s - x \cdot \omega), x),\ p = 0, 1, \ldots. \end{aligned} \tag{1.37}$$

The only difference between (1.37) and (1.18), apart from the fact that there are two terms with an obvious sign change, is that V is allowed to be t-dependent.

The initial conditions in (1.33) translate to initial conditions on the Taylor series. The smoothness of $u|_{t=t_0}$ just corresponds to the absence of jumps in the derivatives. Thus

$$\begin{gathered} u|_{t=t_0} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1}) \Longleftrightarrow \\ g_+(t_0, x, t_0, s, \omega) \equiv g_-(t_0, x, t_0, s, \omega) \\ \text{in Taylor series at } x \cdot \omega = s. \end{gathered} \tag{1.38}$$

That is,

$$\begin{gathered} g_{+,k}(x, t_0, s, \omega) = (-1)^k g_{-,k}(x, t_0, s, \omega) \\ \text{on } x \cdot \omega = s \ \forall\ k = 0, 1, \dots. \end{gathered} \tag{1.39}$$

Similarly the condition on the first derivative in (1.33) becomes

$$\begin{gathered} g_{+,0}(x, t_0, s, \omega) + g_{-,0}(x, t_0, s, \omega) = i \text{ on } x \cdot \omega = s \\ g_{+,k}(x, t_0, s, \omega) + (-1)^k g_{-,k}(x, t_0, s, \omega) = 0 \\ \text{on } x \cdot \omega = s \ \forall\ k = 1, 2, \dots. \end{gathered} \tag{1.40}$$

Combining these two sets of conditions gives

$$\begin{gathered} g_{\pm,0} = \frac{i}{2}, \\ g_{\pm,k} = 0 \text{ on } x \cdot \omega = s \text{ for } k = 1, 2, \dots. \end{gathered} \tag{1.41}$$

With these initial conditions the equations (1.36) have unique solutions

$$\begin{aligned} g_{\pm,0} &= \frac{i}{2}, \\ g_{\pm,1} &= \frac{i}{4} \int_s^{x \cdot \omega} V(t_0 \pm (s - r), x + (r - x \cdot \omega)\omega) dr \\ g_{\pm,k} &= \frac{i}{2} \int_s^{x \cdot \omega} b_{k,\pm}(x + (r - x \cdot \omega), s, \omega) dr, \ k > 1 \\ b_{\pm,k}(x, s, \omega) &= \Delta g_{\pm,k-1}(x, s, \omega) \\ &\quad + \sum_{\ell \le k-1} V_{\pm,k-1-\ell}\, g_{\pm,\ell}(x, s, \omega). \end{aligned} \tag{1.42}$$

Summing these Taylor series as before gives a solution to the initial value problem (1.33) with $\mathcal{C}^\infty$ error terms in all variables.

Proposition 1.2. *For $V \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ there exist $g_\pm, h \in \mathcal{C}^\infty(\mathbb{R}^{n+3} \times \mathbb{S}^{n-1})$ such that*

$$\begin{gathered} u(t, x, t_0, s, \omega) = H(t - t_0 + s - x \cdot \omega) g_+(t, x, t_0, s, \omega) \\ + H(t - t_0 - s + x \cdot \omega) g_-(t, x, t_0, s, \omega) + h(t, x, t_0, s, \omega) \end{gathered} \tag{1.43}$$

satisfies the strengthened form of (1.33)*:*

(1.44)
$$\begin{gathered} P_V u = f \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_{t_0} \times \mathbb{R}_s \times \mathbb{S}^{n-1}) \\ u|_{t=t_0} = 0,\ D_t u|_{t=t_0} = \delta(s - x\cdot\omega) \\ \textit{with } D^k_t f(t_0, x, t_0, s, \omega) = 0\ \forall\ k = 0, 1, \dots. \end{gathered}$$

Proof. By adding the $\mathcal{C}^\infty$ term h we need to remove the $\mathcal{C}^\infty$ errors in the initial values of u and the Taylor series of $P_V u$ at $t = t_0$. This in fact just fixes the Taylor series of h at $t = t_0$. Thus if u' is the solution to (1.33) just constructed and $h_k = D^k_t h$ at $t = t_0$ we simply need to choose

(1.45)
$$h_0 = u'|_{t=t_0},\ h_1 = D_t u'|_{t=t_0} - \delta(s - x\cdot\omega),$$

and

(1.46)
$$\begin{gathered} h_{k+2} = -D^k_t[P_V u']|_{t=t_0} + \Delta h_k + \sum_{\ell\le k} \binom{k}{\ell} V_{k-\ell} h_\ell\ \forall\ k \ge 0 \\ \text{where } V_\ell = D^\ell_t V|_{t=t_0}. \end{gathered}$$

□

We shall show below that for (1.1), and similarly for the initial values problem, there is a solution and it differs from the solution with $\mathcal{C}^\infty$ error terms we have constructed by a $\mathcal{C}^\infty$ function, just as one might expect. To remove the $\mathcal{C}^\infty$ error terms in the continuation problem and actually solve (1.1), or similarly to remove the errors in the initial value problem (1.44), we need to find a forward fundamental solution for the perturbed wave operator. We shall look for a distribution $E(t, x; t', x') \in \mathcal{C}^\infty(\mathbb{R}^{2n+2})$ such that

(1.47)
$$\begin{aligned} &P_V E(t, x; t', x') = 0 \\ &E(t', x; t', x') = 0 \\ &D_t E(t', x; t', x') = \delta(x - x'). \end{aligned}$$

Of course there is in principle a problem with the meaning of the restrictions of the distribution to the surface $t = t'$, however this will resolve itself during the construction. The main idea is to produce a close approximation to E (a parametrix for the Cauchy problem) by the superposition of the solutions to (1.44) and then to remove the error term by iteration. The first step in this is to find a way of writing the delta distribution in (1.47) in terms of the plane wave delta distributions in (1.44). To do so we introduce the Radon transform.

CHAPTER 2

Radon transform

As noted above, to construct a parametrix for the Cauchy problem, we need to decompose the Dirac delta function into plane waves. We shall use the Radon transform to do this.

For each $s \in \mathbb{R}$ and direction $\omega \in \mathbb{S}^{n-1}$ consider the hyperplane in $\mathbb{R}^n$ given by

$$HS(s,\omega) = \{x \in \mathbb{R}^n; x \cdot \omega = s\} \tag{2.1}$$

with normal $\omega \in \mathbb{S}^{n-1}$ and distance $|s|$ to the origin. On $HS(s,\omega)$ we define the measure dH_x so that

$$dH_x \wedge ds = dx. \tag{2.2}$$

If we choose j, with $1 \le j \le n$ so that $\omega_j \neq 0$ then $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ give global coordinates in $HS(s,\omega)$ and

$$dH_x = |\omega_j|^{-1}|dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n|. \tag{2.3}$$

Proposition 2.3. *The Radon transform, defined by*

$$Rf(s,\omega) = \int\limits_{HS(s,\omega)} f(x) dH_x. \tag{2.4}$$

is a continuous linear map

$$R : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1}). \tag{2.5}$$

Proof. This follows directly from the fact that the integral in (2.4) is over a bounded domain which varies smoothly with s and ω. □

The Schwartz kernel of R, which we again write as R, is simply

$$R(s,\omega,x) = \delta(s - x \cdot \omega). \tag{2.6}$$

This is closely related to distributions we have been already been dealing with. We proceed to compute the formal transpose of R. Suppose $g \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then

$$\int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} (\int\limits_{x\cdot\omega = s} f(x) dH_x) g(s,\omega) ds dw = \int \int\limits_{\mathbb{S}^{n-1}} f(x) g(x \cdot \omega, \omega) d\omega dx \tag{2.7}$$

where $d\omega$ is the standard measure on the sphere $\mathbb{S}^{n-1}$. From (2.6) it follows that

$$R^t : \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \tag{2.8}$$

is given by

$$R^t g(x) = \int\limits_{\mathbb{S}^{n-1}} g(x \cdot \omega, \omega) d\omega. \tag{2.9}$$

Notice that for every $g \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$, R^tg is a superposition of plane waves. Obviously R^t can be extended as a continuous linear map

$$R^t : \mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \tag{2.10}$$

since the integral in (2.9) is over a compact domain. It is not the case that R^tg always has compact support whenever $g \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$. However if g vanishes near $s = 0$ then R^tg vanishes near the origin:

Proposition 2.4. *Suppose* $g \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ *and* $g = 0$ *for* $|s| \le R$ *then* $R^tg(x) = 0$ *for* $|x| \le R$.

Proof. From (2.9) the integrand vanishes in $|x| \le R$, since if $\omega \in \mathbb{S}^{n-1}$ then $|x\cdot\omega| \le |x| \le R$. □

The Schwartz kernel of R^t is

$$R^t(x, s, \omega) = \delta(s - x\cdot\omega), \tag{2.11}$$

which is the same distribution as the Schwartz kernel of R with the order of the variables changed. Using (2.5) and (2.10) we can extend R and R^t to spaces of distributions by duality.

Definition 2.1. The action of the Radon transform on compactly-supported distributions $R : \mathcal{C}_c^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1})$ is defined by $R(u)(\phi) = u(R^t\phi)$ for $\phi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ and similarly the action of the transpose Radon transform $R^t : \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n)$ is defined by $R^tu(\phi) = u(R\phi)$ where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

Both R and R^t are continuous, linear maps extending (2.5) and (2.10) respectively.

The Radon transform is closely related to the Fourier transform and properties of the latter give very useful properties of the former. Let $\mathcal{F}_s(g)$ denote Fourier transform of $g \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ in the s-variable and let $\hat{f}$ be the full n-dimensional Fourier transform of $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$:

$$\mathcal{F}_s(g)(\sigma, \omega) = \int e^{-is\sigma} g(s, \omega)ds \tag{2.12}$$

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx. \tag{2.13}$$

Proposition 2.5. *For any* $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\mathcal{F}_sRf(\rho, \omega) = \hat{f}(\rho\omega),\ \ \rho \in \mathbb{R}, \omega \in \mathbb{S}^{n-1}. \tag{2.14}$$

Proof. Using the definition of the Radon transform we find

$$\mathcal{F}_s(Rf)(\rho, \omega) = \int_{\mathbb{R}} e^{-is\rho} \int_{x\cdot\omega = s} f(x)dH_x ds. \tag{2.15}$$

Interchanging the variables and using $dH_x \wedge ds = dx$ the identity (2.14) follows since

$$\mathcal{F}_s(Rf)(\rho, \omega) = \int e^{-i(x\cdot\omega)\rho} f(x)dx. \tag{2.16}$$

□

As a very useful consequence of Proposition 2.5 we deduce next that the Radon transform intertwines the n-dimensional and the one- dimensional Laplacians.

PROPOSITION 2.6. *For any* $n \geq 2$

$$R\Delta f = D_s^2 Rf \quad \forall\, f \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{2.17}$$

PROOF. Taking the partial Fourier transform in the s-variable of $R\Delta f$, and using (2.14) we get

$$\mathcal{F}_s(R\Delta f)(\rho,\omega) = \widehat{\Delta f}(\rho\omega) = \rho^2 \hat{f}(\rho\omega). \tag{2.18}$$

Also

$$\mathcal{F}_s(D_s^2 Rf)(\rho,\omega) = \rho^2 \mathcal{F}_s(Rf)(\rho,\omega) = \rho^2 \hat{f}(\rho\omega) \tag{2.19}$$

givin (2.17). □

We shall now prove the Radon inversion formula that allows us to write any compactly supported function (or distribution) as a superposition of plane waves.

THEOREM 2.1. *For any* $n \geq 2$ *and any* $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$f = \frac{1}{2(2\pi)^{n-1}} R^t(|D_s|^{n-1} Rf) \tag{2.20}$$

where

$$|D_s|^{n-1} g = \begin{cases} D_s^{n-1} g & \text{for } n \text{ odd and} \\ \frac{1}{2\pi}\int_0^\infty e^{is\rho}\hat{g}(\rho\omega)|\rho|^{n-1} d\rho & \text{for } n \text{ even} \end{cases} \tag{2.21}$$

for any $g \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$.

PROOF. The Fourier inversion formula is

$$f(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{f}(\xi) d\xi. \tag{2.22}$$

Changing to polar coordinates gives

$$f(x) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\rho x\cdot\omega} \hat{f}(\rho\omega)\rho^{n-1} d\omega d\rho \tag{2.23}$$

so

$$f(x) = \frac{1}{(2\pi)^n} R^t g(x) \tag{2.24}$$

where $g(s,\omega) = \int_0^\infty e^{is\rho}\hat{f}(\rho\omega)\rho^{n-1}d\rho$. In the case that n is odd we can write

$$g(s,\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{is\rho}\hat{f}(\rho\omega)\rho^{n-1}d\rho = \frac{1}{2}\mathcal{F}_s^{-1}[D_s^{n-1}\mathcal{F}_s(Rf)] = \frac{1}{2}D_s^{n-1}Rf \tag{2.25}$$

where we have used the fact that $Rf(-s,-\omega) = Rf(s,\omega)$. This proves the theorem in that case. For n even

$$g(s,\omega) = \frac{1}{2}|D_s|^{n-1}Rf \text{ by definition.} \tag{2.26}$$

□

The Radon transform extends by continuity to a linear map on the Schwartz spaces of rapidly decreasing $\mathcal{C}^\infty$ functions

$$R : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1}) \tag{2.27}$$

and hence its transpose extends to tempered distributions:

$$R^t : \mathcal{S}'(\mathbb{R} \times \mathbb{S}^{n-1}) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{2.28}$$

The continuity ensures that Proposition 2.5, Theorem 2.6 and Proposition 2.1 are valid in these spaces. It is not the case that R^t maps the space $\mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1})$ into $\mathcal{S}(\mathbb{R}^n)$. We shall examine the structure of $R^t[\mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1})]$ below.

We also note the Plancherel formula for the Radon transform.

Lemma 2.2. *For any $f_1, f_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$*

$$\int\limits_{\mathbb{R}^n} f_1(x)\overline{f_2(x)}dx = \tag{2.29}$$

$$\frac{1}{2}\frac{1}{(2\pi)^{n-1}} \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} (|D_s|^{\frac{n-1}{2}} Rf_1)(s,\omega)\overline{(|D_s|^{\frac{n-1}{2}} Rf_2)(s,\omega)}dsd\omega. \tag{2.30}$$

Proof. The Plancherel formula for the Fourier transform gives

$$\int\limits_{\mathbb{R}^n} f_1(x)\overline{f_2(x)}dx = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{R}^n} \widehat{f_1}(\xi)\overline{\widehat{f_2}(\xi)}d\xi. \tag{2.31}$$

Introducing polar coordinates as above the right side of (2.31) can be written

$$\frac{1}{2}\frac{1}{(2\pi)^{n}} \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} \mathcal{F}_s(|D_s|^{\frac{n-1}{2}} Rf_1)(s,\omega)\overline{\mathcal{F}_s(|D_s|^{\frac{n-1}{2}} Rf_2)(s,\omega)}dsd\omega. \tag{2.32}$$

Using the one-dimensional Plancherel formula this reduces to (2.30). □

Using these results we can now express the delta function as a superposition of plane waves.

Proposition 2.7. *For n odd*

$$\delta_0 = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \delta^{n-1}(x\cdot\omega)d\omega. \tag{2.33}$$

For n even

$$\delta_0 = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} |\delta|^{n-1}(x\cdot\omega)d\omega \tag{2.34}$$

where

$$\delta^{n-1}(x\cdot\omega) = \lim_{s\to 0} \partial_s^{n-1}\delta(s + x\cdot\omega) \tag{2.35}$$

and

$$|\delta|^{n-1}(x\cdot\omega) = \lim_{s\to 0} |\partial_s|^{n-1}\delta(s + x\cdot\omega). \tag{2.36}$$

PROOF. This follows directly from the Fourier inversion formula. For n odd

$$\begin{aligned}\delta_0(\phi) = \phi(0) &= \frac{1}{2(2\pi)^{n-1}} R^t \partial_s^{n-1} R\phi(0) \\ &= \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} \partial_s^{n-1} R\phi(0,\omega) d\omega\end{aligned} \tag{2.37}$$

and similarly for n even. □

Although this is the main use we shall make of the Radon transform, at least for the moment, we note a few other facts. The inversion formula shows that as a map (2.5) R is injective. It is however *not* surjective. To see this observe that if $\psi = R(\phi)$ with $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}} s^p \psi(s,\omega) ds = \int_{\mathbb{R}^n} (x \cdot \omega)^p \phi(x) dx \tag{2.38}$$

is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous polynomial of degree p for all $p \in \mathbb{N}$. This is certainly not the case for an arbitrary function $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$.

Conversely we wish to show that these conditions characterize the range of R on $\mathcal{C}_c^\infty(\mathbb{R}^n)$. In fact this is a result due to Helgason [**1**].

PROPOSITION 2.8. *(Helgason) For any $n \geq 3$, odd, the range of the Radon transform on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ consists precisely of the subspace of those $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ which are invariant under the reflection $(s,\omega) \to (-s,-\omega)$ and are such that for every $p \in \mathbb{N}$ there is a homogeneous polynomial, Q_p, of degree p on $\mathbb{R}^n$ with*

$$\int_{\mathbb{R}} s^p \psi(s,\omega) ds = Q_p(\omega). \tag{2.39}$$

PROOF. Since we know that

$$\mathcal{F}_s(R\phi)(\sigma,\omega) = \hat{\phi}(\sigma\omega) \tag{2.40}$$

consider the function

$$f(\sigma\omega) = \int_{\mathbb{R}} e^{-is\sigma} \psi(s,\omega) ds. \tag{2.41}$$

where $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$. Initially we define f for $\sigma \neq 0$. It is certainly $\mathcal{C}^\infty$ in $\sigma \neq 0$, and directly from (2.41) it is bounded near $\sigma = 0$, so is a well-defined distribution on $\mathbb{R}^n$ with variable $\xi = \sigma\omega$. The conditions (2.38) ensure that it is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Indeed only the smoothness at $\xi = 0$ needs to be checked. Replacing the exponential by its Taylor series to high order:

$$e^{-is\sigma} = \sum_{j \leq k} \frac{(-is\sigma)^k}{k!} + (s\sigma)^k \mu_k(s\sigma) \tag{2.42}$$

notice that the smooth coefficient in the remainder term satisfies

$$|\partial_r^m \mu_k(r)| \leq C_{k,m}(1+|r|)^m \ \forall\ m. \tag{2.43}$$

Inserting this expansion into (2.41) gives an expansion for f near $\xi = 0$. The terms arising from the sum in (2.42) are each of the form $Q_k(\omega)\sigma^k = Q_k(\xi)$ where Q_k is a homogeneous polynomial. Thus to prove the regularity of f at $\xi = 0$ it is only necessary to check that the remainder terms become increasingly smooth in a neighbourhood of 0. In the polar coordinates (σ, ω) the remainder is $\mathcal{C}^\infty$ and vanishes to order k at $\sigma = 0$. It follows that it is $(k-1)$ times differentiable (at least) in ξ near 0, hence $\mathcal{C}^\infty$.

Thus we have shown that if f is defined by (2.41) then $f \in \mathcal{S}(\mathbb{R}^n)$. This implies in particular that $f = \hat{\phi}$ for some $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi = R(\phi)$. It remains to show that $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Recalling that we have assumed n to be odd set

$$\psi' = \frac{1}{2}(2\pi)^{-n} D_s^{n-1}\psi \text{ so } \phi = R^t(\psi'),\ \psi' \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1}). \tag{2.44}$$

We can write this explicitly as

$$\phi(r\theta) = \int\limits_{\mathbb{S}^{n-1}} \psi'(r\theta \cdot \omega, \omega) d\omega. \tag{2.45}$$

Now the support of $\psi'(s, \omega)$ is confined to some compact region $|s| \le \rho$. Thus the integral in (2.45) is actually limited to the region $|\theta \cdot \omega| \le \rho/r$ which is a small neighbourhood of an equatorial $\mathbb{S}^{n-2}$ as $r \to \infty$. Let $h_k \in \mathcal{C}^\infty(\mathbb{R}^n)$ be any polynomial homogeneous of degree k. Averaging over the sphere gives

$$\int\limits_{\mathbb{S}^{n-1}} h_k(\theta)\phi(r\theta) d\theta \tag{2.46}$$

$$= \int\limits_{\mathbb{R}\times\mathbb{S}^{n-2}\times\mathbb{S}^{n-1}} h_k(st\omega + \theta')\psi'(s, \omega)(1 - (st)^2)^{\frac{n-3}{2}} t ds d\theta' d\omega,\ rt = 1. \tag{2.47}$$

Here we have introduced the change of variables

$$\theta = (\theta \cdot \omega)\omega + (1 - (\theta \cdot \omega)^2)^{\frac{1}{2}}\theta' \tag{2.48}$$

and then set $\omega \cdot \theta = st$. The important point is that, for $|t| < \rho/r$, this shows the average to be real-analytic in t, since h_k is a polynomial and the other t-dependence is clearly analytic on the support of ψ'. However, by the discussion above we know that as $t \to 0$ the function ϕ is rapidly decreasing. Thus the Taylor series of (2.47) in t at $t = 0$ vanishes. It follows that the average over the sphere vanishes in $r > \rho$ for every homogeneous polynomial h_k. The completeness properties of spherical harmonics show that this means that

$$\phi(x) = 0 \text{ in } |x| > \rho \text{ if } \psi = 0 \text{ in } |s| > \rho. \tag{2.49}$$

□

Exercise 2.1. Notice that if n is even then (2.44) will not lead to (2.45). Replacing D_s^{n-1} by $|D_s|^{n-1}$ does lead to (2.45), but ψ' no longer has compact support in s. Try to finish the proof of Proposition 2.8 in case n is even.

We note one immediate consequence of the first part of the proof of Proposition 2.8, where the parity of n was not used.

Corollary 2.1. *For any $n \ge 2$ the image $R[\mathcal{S}(\mathbb{R}^n)]$ of the Schwartz space under the Radon transform is the subspace of $\mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1})$ consisting of those functions even under $(s, \omega) \longrightarrow (-s, -\omega)$ and such that for each $p \in \mathbb{N}$ there is a homogeneous polynomial Q_p of degree p on $\mathbb{R}^n$ satisfying (2.39).*

Next we consider the image of the Schwartz space under the transpose R^t. From (2.10) we know that $R^t\left[\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})\right]$ consists of $\mathcal{C}^\infty$ functions. Its element are arbitrary smooth functions in the interior but have asymptotic expansions at infinity. To understand this in a direct way we shall compactify $\mathbb{R}^n$ to a ball. The map

$$Q:\mathbb{R}^n\ni x\longmapsto y=\frac{2}{\pi}\arctan|x|\cdot\frac{x}{|x|}\in\mathbb{B}^n=\{y\in\mathbb{R}^n;|y|\le 1\} \tag{2.50}$$

is an invertible $\mathcal{C}^\infty$ map onto the interior of the ball, with inverse just

$$Q^{-1}:\overset{\circ}{\to}\mathbb{B}^n\ni y\longmapsto\tan\frac{\pi|y|}{2}\cdot\frac{y}{|y|}\in\mathbb{R}^n,\ \overset{\circ}{\to}\mathbb{B}^n=\{y\in\mathbb{B}^n;|y|<1\}. \tag{2.51}$$

Thus any element of $\mathcal{C}^\infty(\mathbb{R}^n)$ defines an element of $\mathcal{C}^\infty(\overset{\circ}{\to}\mathbb{B}^n)$ by

$$\mathcal{C}^\infty(\mathbb{R}^n)\ni f\longmapsto(Q^{-1})^*f=f\cdot Q^{-1}\in\mathcal{C}^\infty(\overset{\circ}{\to}\mathbb{B}^n). \tag{2.52}$$

It is important to distinguish between $\mathcal{C}^\infty(\overset{\circ}{\to}\mathbb{B}^n)$, the space of all smooth functions on the interior of the ball, and $\mathcal{C}^\infty(\mathbb{B}^n)$, the space of such smooth functions with all derivatives continuous up to the bounding sphere. Of course

$$\mathcal{C}^\infty(\mathbb{B}^n)\subset\mathcal{C}^\infty(\overset{\circ}{\to}\mathbb{B}^n). \tag{2.53}$$

We consider also the subspace

$$\mathcal{C}^\infty_0(\mathbb{B}^n)=\{u\in\mathcal{C}^\infty(\mathbb{B}^n);u=0\text{ on the boundary.}\} \tag{2.54}$$

Lemma 2.3. *For any $n\ge 2$*

$$(Q^{-1})^*R^t:\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})\longrightarrow\mathcal{C}^\infty_0(\mathbb{B}^n). \tag{2.55}$$

Proof. By definition

$$R^tg(x)=\int_{\mathbb{S}^{n-1}}g(x\cdot\omega,\omega)d\omega,\ g\in\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1}). \tag{2.56}$$

Set $x=r\theta$ where $\theta\in\mathbb{S}^{n-1}$ and $r=|x|$. Changing coordinates in the integral in (2.56) by setting $\omega=t\theta+(1-t^2)^{\frac12}\omega'$, with $\omega'\cdot\theta=0$ gives

$$R^tg(r\theta)=\int_{\mathbb{S}^{n-2}}\int_{[-1,1]}g(rt,t\theta+(1-t^2)^{\frac12}\omega')(1-t^2)^{\frac{n-3}{2}}dtd\omega'. \tag{2.57}$$

Setting $r=1/\rho$ and $T=rt=t/\rho$ gives

$$R^tg(\theta/\rho)$$
$$=\rho\int_{\mathbb{S}^{n-2}}\int_{[-1/\rho,1/\rho]}g(T,\rho T\theta+(1-\rho^2T^2)^{\frac12}\omega')(1-\rho^2T^2)^{\frac{n-3}{2}}dTd\omega'. \tag{2.58}$$

The integrand is uniformly (in ρ) rapidly decreasing as $T\to\infty$ so, as $\rho\to 0$, the integral in (2.58) converges to

$$\int_{\mathbb{R}\times\theta^\perp}g(T,\omega')dTd\omega'. \tag{2.59}$$

Thus $R^tg(\theta/\rho)$ is the product of ρ and a continuous function of the variables $(\rho,\theta)\in[0,1)\times\mathbb{S}^{n-1}$. From (2.50) and (2.51) this implies in particular that $(Q^{-1})^*R^tg$ is a continuous function on $\mathbb{B}^n$ vanishing on the boundary. To get further regularity we simply differentiate (2.58) with respect to ρ and θ. The angular derivatives are of the same general form. Differentiation with respect to ρ within the integrand also produces terms which can be shown to be continuous in the same way. Differentiation of the limits of integration in (2.58) produces factors of ρ^{-2} but since the integrand is uniformly rapidly decreasing as $T = \pm 1/\rho \to \pm\infty$ these are also continuous. Thus we conclude that $R^t g(\theta/\rho)$ is $\mathcal{C}^\infty$ as a function of $(\rho, \theta) \in [0, 1) \times \mathbb{S}^{n-1}$. Together with the obvious smoothness near 0 this just gives (2.55). □

There are other useful ways to restate this result. In particular $R^t g$ is an example of a 'classical symbol.' This means that it has an expansion near infinity:

$$R^t g(x) \sim \sum_{j \le -1} f_j(x) \text{ as } |x| \to \infty \tag{2.60}$$

where the f_j are $\mathcal{C}^\infty$ functions homogeneous of degree j in $|x| > 0$ and (2.60) means that for each $N \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^n$ there is a constant $C = C_{N,\alpha}$ such that

$$\left| D_x^\alpha \left[R^t g - \sum_{-N \le j \le -1} f_j(x) \right] \right| \le C|x|^{-N-1-|\alpha|} \text{ in } |x| \ge 1. \tag{2.61}$$

In terms of (2.55) this is just the Taylor series expansion of $R^t g(\theta/\rho)$ at $\rho = 1/|x| = 0$, i.e. the boundary of the ball. Spaces of symbols (somewhat more general than these) will play an important rôle below and are a very important component of the microlocal analysis of distributions. We shall exploit Lemma 2.3 to get a further continuous extension of the Radon transform, by duality, after the discussion of the symbol spaces.

If $g \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1})$ then (2.58) gives explicit formulæ for the coefficients in the expansion

$$(Q^{-1})^* R^t g \sim \sum_{k \ge 0} \rho^{1+k-2p} f_k(\theta). \tag{2.62}$$

Namely

$$f_k(\theta) = \sum_{p \le k/2} C_p \int_{\mathbb{R}} \int_{\theta^\perp} T^k (V_{\omega'}^{k-2p} g)(T, \omega') d\omega' dT \tag{2.63}$$

where $V_{\omega'}$ is the unit vector field on the circle spanned by θ and ω'. and the C_p are the coefficients in the Taylor series

$$(1 - t^2)^{\frac{n-3}{2}} = \sum_p C_p t^2 p. \tag{2.64}$$

If $g(-s, -\omega) = g(s, \omega)$ for all $s \in \mathbb{R}$ and $\omega \in \mathbb{S}^{n-1}$ then for all odd integers k the f_k vanish.

Notice the effect, on (2.64) and hence (2.62), of the parity of the dimension. If n is odd then the Jabobian factor $(1 - \rho^2 T^2)^{(n-3)/2}$ is a polynomial and the sum in (2.63) is limited to $p \le n - 3$. From this we can easily recover Corollary 2.1.

Remarks: (1) Probably push these last comments a little further, for use in Chapter 11. (2) Maybe the proof of Proposition 2.43 needs to be clarified a bit. (3) Can the number of variables in (2.58) be reduced? (4) Is the discussion following (2.59) confusing? (5) Clean up (2.47) and (2.58)?

CHAPTER 3

Distributions conormal at a hypersurface

In the construction, in Chapter 1, of progressing wave solutions, modulo $\mathcal{C}^\infty$ errors, to the wave equation with potential we dealt with the class of distributions having only jump discontinuities across a hypersurface. We shall now discuss an important space which includes these and other distributions having the simplest type of singularity, which is generally described as *conormal,* across a hypersurface. The restriction to hypersurfaces conveniently simplifies the discussion but it will be removed later.

In $\mathbb{R}^n$ consider the model hypersurface

$$H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 = 0\}. \tag{3.1}$$

The distributions we consider will have regularity properties similar to $\delta(x_1)$ and $H(x_1)$. Recall that Sobolev spaces can be defined in terms of the Fourier transform:

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}. \tag{3.2}$$

Definition 3.2. The space $I_c^{(s)}(\mathbb{R}^n, H) \subset \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ consists of those distributions of compact support, u, for which

$$(x_1 D_1)^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} u \in H^s(\mathbb{R}^n) \quad \forall\ \alpha \in \mathbb{N}^n. \tag{3.3}$$

Since the Sobolev spaces decrease with increasing order, s, these conormal distributions have similar properties:

$$I_c^{(s)}(\mathbb{R}^n, H) \subset I_c^{(s')}(\mathbb{R}^n, H) \text{ if } s \geq s'. \tag{3.4}$$

Since (3.3) implies in particular that $u \in H^s(\mathbb{R}^n)$ we certainly have

$$I_c^{(\infty)}(\mathbb{R}^n, H) = \bigcap_s I_c^{(s)}(\mathbb{R}^n, H) = \mathcal{C}_c^\infty(\mathbb{R}^n) \tag{3.5}$$

is independent of H. On the other hand it is very important to note that

$$I_c^{(-\infty)}(\mathbb{R}^n, H) = \bigcup_s I_c^{(s)}(\mathbb{R}^n, H) \neq \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{3.6}$$

For example, in $n > 1$, the Dirac delta at 0

$$\delta(x) \notin I^{(-\infty)}(\mathbb{R}^n, H). \tag{3.7}$$

Indeed, if $s \in \mathbb{R}$ and $\ell > -s$ then $D_{x_2}^\ell \delta(x) \notin H^s(\mathbb{R}^n)$. This violates (3.3), so proves (3.7).

One easy consequence of Leibniz' formula for the distribution of differentiation over a product is that each $I_c^{(s)}(\mathbb{R}^n, H)$ is a $\mathcal{C}^\infty(\mathbb{R}^n)$-module. Thus if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$

and $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ then

$$(x_1D_1)^{\alpha_1}D_2^{\alpha_2}\dots D_n^{\alpha_n}(\phi u) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(x_1D_1)^{\beta_1}D_2^{\beta_2}\dots D_n^{\beta_n}\phi \times (x_1D_1)^{\alpha_1-\beta_1}D_2^{\alpha_2-\beta_2}\dots D_n^{\alpha_n-\beta_n}u. \tag{3.8}$$

Then from (3.3) and the fact that $H_c^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)\cap\mathcal{C}_c^{-\infty}\mathbb{R}^n)$ is a $\mathcal{C}^\infty$module it follows that

$$\mathcal{C}^\infty(\mathbb{R}^n)\cdot I_c^{(s)}(\mathbb{R}^n,H) = I_c^{(s)}(\mathbb{R}^n,H) \quad \forall\ s. \tag{3.9}$$

This allows us to define the similar space without restriction on supports:

$$I^{(s)}(\mathbb{R}^n,H) = \{u\in\mathcal{C}^{-\infty}(\mathbb{R}^n); \phi u \in I_c^{(s)}(\mathbb{R}^n,H) \quad \forall\ \phi\in\mathcal{C}_c^\infty(\mathbb{R}^n)\}. \tag{3.10}$$

Then

$$I_c^{(s)}(\mathbb{R}^n,H) = I^{(s)}(\mathbb{R}^n,H)\cap\mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{3.11}$$

Exercise 3.2. The delta 'function' $\delta(x_1)$ is conormal with respect to $\{x_1 = 0\} = H$. Indeed we have

$$x_1\delta(x_1) = 0,\ D_j\delta(x_1) = 0 \text{ if } j\ge 2. \tag{3.12}$$

From the first of these identities it follows that

$$x_1D_1\delta(x_1) = [x_1,D_1]\delta(x_1) + D_1(x_1\delta(x_1)) = i\delta(x_1). \tag{3.13}$$

Thus we conclude

$$(x_1D_1)^{\alpha_1}D_2^{\alpha_2}\dots D_n^{\alpha_n}\delta(x_1) = \begin{cases} 0 & \text{if } \alpha_2+\dots+\alpha_n>0 \\ i^{\alpha_1}\delta(x_1) & \text{otherwise.}\end{cases} \tag{3.14}$$

More generally suppose $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$, then

$$\phi\delta(x_1) = \phi(0,x')\delta(x_1) \tag{3.15}$$

so from (3.14)

$$(x_1D_1)^{\alpha_1}D_2^{\alpha_2}\dots D_n^{\alpha_n}[\phi\delta(x_1)] = (D')^{\alpha'}\phi(0,x')i^{-\alpha_1}\delta(x_1)\in H_c^{-1}(\mathbb{R}^n). \tag{3.16}$$

In particular $\delta(x_1)\in I^{(-1)}(\mathbb{R}^n,H)$.

Exercise 3.3. Show that

$$(x_1)_+^j = H(x_1)x_1^j \in I^{(j)}(\mathbb{R}^n,H)\ \forall\ j\in\mathbb{N}. \tag{3.17}$$

Observe that for any $l\in\mathbb{N}$ there are constants $C_{\ell,j}$ such that

$$(x_1D_1)^l = \sum_{j\le l} C_{l,j}x_1^jD_1^j \qquad C_{l,l}=1. \tag{3.18}$$

Indeed this follows easily by induction as we proceed to show. It is surely true for $\ell = 1$. Applying x_1D_1 to both sides of (3.18) gives the identities

$$C_{l+1,j+1} = C_{l,j} - iC_{l,j+1} \tag{3.19}$$

provided we set $C_{l,k}=0$ if $k>l$, $C_{1,1}=1$. This proves (3.18).

The sub-diagonal nature of (3.18) means that it can be inverted, so that

$$x_1^jD_1^j = \sum_{l\le j} d_{j,l}(x_1D_1)^l \quad \forall\ j>0. \tag{3.20}$$

This means that the conditions (3.3) are equivalent to

$$x_1^{\alpha_1} D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} u \in H^s(\mathbb{R}^n), \tag{3.21}$$

if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$. One immediate consequence of this is the regularity away from H :

Lemma 3.4. *If* $u \in I^{(s)}(\mathbb{R}^n, H)$ *then*

$$\operatorname{singsupp}(u) \subset H. \tag{3.22}$$

Proof. If $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has $\operatorname{supp}(\phi) \cap H = \emptyset$ then ϕu satisfies (3.3), and hence (3.21), and on $\operatorname{supp}(\phi)$ $x_1 \neq 0$. Thus (3.21) implies $D^\alpha(\phi u) \in H^s(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n$. This implies $\phi u \in \mathcal{C}^\infty(\mathbb{R}^n)$, proving (3.22). □

It should not be thought that (3.22) characterizes conormal distributions. For example $\delta(x)$ satisfies (3.22) but is not conormal with respect to $H = \{x_1 = 0\}$ – see (3.7).

Next consider the action of differential operators. If

$$P = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha \tag{3.23}$$

is a linear differential operator of order m with $\mathcal{C}^\infty$ coefficients then

$$P : I^{(s)}(\mathbb{R}^n, H) \longrightarrow I^{(s-m)}(\mathbb{R}^n, H) \quad \forall\ s. \tag{3.24}$$

In view of (3.9) it is enough to check this for $P = D^\alpha$. In fact it is enough to take $|\alpha| = 1$ and then iterate. Since D_j, for $j > 2$, commutes with the operator $(x_1 D_1)^{\alpha_1} (D_1')^{\alpha'}$ only the case $\alpha = (1, 0, \dots 0)$ is not immediate. In the form (3.21) this is equally obvious since

$$x_1^{\alpha_1} D_1^{\alpha_1} (D')^{\alpha'} (D_1 u) = D_1 [x^{\alpha_1} D_1^{\alpha_1} (D')^{\alpha'}] u + i\alpha_1 x^{\alpha_1 - 1} D_1^{\alpha_1 - 1} (D')^{\alpha'} (D_1 u) \tag{3.25}$$

so the same regularity $D_1 u \in H^{s-1}(\mathbb{R}^n)$ follows by induction on α_1. Thus we have proved (3.24).

Perhaps the most important aspects of conormal distributions are their *symbolic* properties. These show that they behave in a manner very similar to functions having jump discontinuities. For functions with jump discontinuities across the hypersurface H the symbol, as we shall see later, reduces (in essence) to the smooth function on the hypersurface which is the difference of the limits from the two sides.

We shall define the symbol in general by considering the one-dimensional Fourier transform in a direction across the front. This is well defined for any distribution of compact support:

$$\tilde{u}(\xi, x') = \int e^{-ix_1\xi} u(x_1, x') dx_1 \quad u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{3.26}$$

Proposition 3.9. *For each* $s \in \mathbb{R}$ *there exists* $M(> -s - \frac{1}{2})$ *such that if* $u \in I^{(s)}(\mathbb{R}^n, H)$ *has compact support then* $\tilde{u}(\xi, x') \in \mathcal{C}^\infty(\mathbb{R}^n)$ *satisfies the (symbolic) estimates*

$$\left| D_\xi^\ell (D_{x'}')^{\alpha'} \tilde{u}(\xi, x') \right| \le C_{\alpha', \ell} (1 + |\xi|)^{M - \ell},\ \forall\ \ell \in \mathbb{N},\ \alpha \in \mathbb{N}^{n-1}. \tag{3.27}$$

PROOF. We shall first take the Fourier transform in all variables and then invert in the dual to the x' variables. Since $u \in H^s(\mathbb{R}^n)$ it follows that with

$$\begin{gathered}\hat{u}(\xi,\eta) = \int_{\mathbb{R}^n} e^{-ix_1\xi - ix'.\eta} u(x_1,x')dx_1dx' \\ \hat{u}(\xi,\eta) = (1+|\xi|+|\eta|)^{-s}g, \ g \in L^2(\mathbb{R}_\xi \times \mathbb{R}^{n-1}_\eta).\end{gathered} \tag{3.28}$$

The argument by which we arrived at (3.21) shows just as well that (3.3) is equivalent to

$$D_1^{\alpha_1}(D'_{x'})^{\alpha'}(x_1^{\alpha_1}u) \in H^s(\mathbb{R}^n) \quad \forall \ \alpha \in \mathbb{N}^n. \tag{3.29}$$

Taking the Fourier transform we find that

$$\xi^{\alpha_1}\eta^{\alpha'}(D_\xi^{\alpha_1}\hat{u}) = (1+|\xi|+|\eta|)^{-s}g_\alpha, \ g_\alpha \in L^2(\mathbb{R}^n) \ \forall \ \alpha. \tag{3.30}$$

Since the multiindex α' is arbitrary this shows that for each $\ell \in \mathbb{N}$

$$\xi^\ell D_\xi^\ell \hat{u}(\xi,\eta) = (1+|\xi|)^{-s}(1+|\eta|)^{-n}g_l(\xi,\eta), \ g_l \in L^2(\mathbb{R}^n). \tag{3.31}$$

Using the same estimates for x_1u we quickly conclude that

$$\begin{cases} D_\xi\hat{u} = (1+|\xi|)^{-s-1}(1+|\eta|)^{-n}g'(\xi,\eta) \\ \hat{u} = (1+|\xi|)^{-s}(1+|\eta|)^{-n}g''(\xi,\eta) \end{cases} \tag{3.32}$$

where $g', g'' \in L^2(\mathbb{R}^n)$. Integrating in ξ (to show continuity) and taking the inverse Fourier transform in η shows that

$$|\tilde{u}(\xi,x')| \le C(1+|\xi|)^{-s}. \tag{3.33}$$

This is the first estimate in (3.27) – for $\ell = 0, \alpha' = 0$. However notice that

$$u \in I_c^{(s)}(\mathbb{R}^n, H) \Longrightarrow D_1^{\alpha_1}x_1^{\alpha_1}D_{x'}^{\alpha'}u \in I_c^{(s)}(\mathbb{R}^n, H). \tag{3.34}$$

Thus (3.33) applies to $D_1^{\alpha_1}x_1^{\alpha_1}D_{x'}^{\alpha'}u$ for all α, so

$$|\xi^\ell D_\xi^\ell D_{x'}^{\alpha'}u(\xi,x')| \le C_{l,\alpha'}(1+|\xi|)^{-s} \quad \forall \ \ell, \alpha'. \tag{3.35}$$

These estimates give (3.27), proving the Proposition although with $M = -s$ rather than for any $M > -s - \frac{1}{2}$ as claimed.

The better estimate follows from (3.32) which can now be improved to

$$\begin{gathered} D_\xi\tilde{u}(\xi,x') = (1+|\xi|)^{-s-1}h'(\xi,x') \\ \tilde{u}(\xi,x') = (1+|\xi|)^{-s}h(\xi,x') \\ \sup_{x'} \int_{\mathbb{R}} [|h'(\xi,x')|^2 + |h(\xi,x')|^2]d\xi < \infty \end{gathered} \tag{3.36}$$

If $s > -\frac{1}{2}$ then integration in from infinity gives

$$|u(\xi,x')| = |\int_\xi^\infty D_\xi u(\xi',x')d\xi'| \le C(1+|\xi|)^{-s-\frac{1}{2}} \text{ as } \xi \to \infty \tag{3.37}$$

and similarly as $\xi \to -\infty$. If $s < -\frac{1}{2}$ integration from 0 gives a similar estimate

$$|u(\xi,x')| \le |u(0,x')| + |\int_0^\xi D_\xi u(\xi',x')d\xi'| \le C(1+|\xi|)^{-s-\frac{1}{2}}. \tag{3.38}$$

In the borderline case of $s = -\frac{1}{2}$ the latter argument still applies but the possibility of logarithmic growth occurs. Thus in this case only the best power law is

$$|u(\xi, x')| \le C(1 + |\xi|)^{-s-\frac{1}{2}+\epsilon} \ \forall \ \epsilon > 0. \tag{3.39}$$

This completes the proof of the proposition. □

We now reverse the arguments above and define a closely related space, which gives the precise 'order' of a conormal distribution.

Definition 3.3. For any $m \in \mathbb{R}$ the space of conormal distributions of order m associated to the hypersurface H in (3.1), $I^{m-\frac{n}{4}+\frac{1}{2}}(\mathbb{R}^n, H) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$, consists of those distributions, u, such that for every $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ the partial Fourier transform of ϕu is $\mathcal{C}^\infty$ and

$$|D^\ell_\xi D^{\alpha'}_{x'} \widetilde{\phi u}(\xi, x')| \le C_{\ell,\alpha'}(1 + |\xi|)^{m-\ell} \quad \forall \ \ell, \alpha'. \tag{3.40}$$

In general we shall denote the *symbol space* corresponding to (3.40) as $S^m(\mathbb{R}^{m-1}, \mathbb{R})$:

$$\begin{gathered} a \in S^m(\mathbb{R}^{n-1}, \mathbb{R}) \Longleftrightarrow \\ a \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \forall \ K \subset\subset \mathbb{R}^{n-1}, \ \ell \in \mathbb{N}, \ \alpha' \in \mathbb{N}^{n-1} \ \exists \ C_{K,l,\alpha'} \text{ s.t.} \\ |D^l_\xi D^{\alpha'}_{x'} a(\xi, x')| \le C_{K,\ell,\alpha'}(1 + |\xi|)^{m-l} \text{ for } (\xi, x') \in \mathbb{R} \times K. \end{gathered} \tag{3.41}$$

In (3.40) we do not have to include the compact set K since the support is compact in x! One might well ask why the strange normalization $m - \frac{n}{4} + \frac{1}{2}$ is used, rather than just m. This should become clear later, in any case it is only a matter of 'convenience.'

Next we check that we have really defined the same space except for a different 'filtration.'

Proposition 3.10. *For every* $m \in \mathbb{R}$

$$I^{m-\frac{n}{4}+\frac{1}{2}}(\mathbb{R}^n, H) \subset I^{(-m-\frac{1}{2}-\epsilon)}(\mathbb{R}^n, H) \subset I^{m-\frac{n}{4}+\frac{1}{2}-\delta}(\mathbb{R}^n, H) \ \textit{if} \ \delta > \epsilon > 0. \tag{3.42}$$

Proof. The first inclusion is easy since for any $\epsilon > 0$

$$(1 + |\xi|)^{-\frac{1}{2}-\epsilon} L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}). \tag{3.43}$$

Thus we find that

$$I^{m-\frac{n}{4}+\frac{1}{2}}_c(\mathbb{R}^n, H) \subset H^s(\mathbb{R}^n) \text{ if } s < -m - \frac{1}{2}. \tag{3.44}$$

Here $I^m_c(\mathbb{R}^n, H) = I^m(\mathbb{R}^n, H) \cap \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$. Now

$$x_1 D_1, D_j : I^m(\mathbb{R}^n, H) \longrightarrow I^m(\mathbb{R}^n, H) \quad \forall \ m, \ j \ge 1 \tag{3.45}$$

so from (3.44) it follows that the estimates (3.3) hold, giving the first part of (3.42). The second part follows from Proposition 3.9 above. □

The inclusions (3.42) are equivalent to

$$I^{-s-\frac{n}{4}-\epsilon}_c(\mathbb{R}^n, H) \subset I^{(s)}_c(\mathbb{R}, H) \subset I^{-s-\frac{n}{4}+\epsilon}_c(\mathbb{R}^n, H) \tag{3.46}$$

$$I^{(-m-\frac{n}{4}+\epsilon)}_c(\mathbb{R}^n, H) \subset I^m_c(\mathbb{R}^n, H) \subset I^{(-m-\frac{n}{4}-\epsilon)}_c(\mathbb{R}^n, H). \tag{3.47}$$

We shall use the sandwiching of the spaces in (3.47) to prove the coordinate invariance of the $I^m(\mathbb{R}^n, H)$. Recall that if $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ then

$$F^* : H^s_c(\mathbb{R}^n) \longrightarrow H^s_c(\mathbb{R}^n) \quad \forall \ s. \tag{3.48}$$

LEMMA 3.5. *If F is a diffeomorphism of $\mathbb{R}^n$ which maps H into itself then*

$$F^* : I^{(s)}(\mathbb{R}^n, H) \longrightarrow I^{(s)}(\mathbb{R}^n, H) \quad \forall\ s. \tag{3.49}$$

PROOF. This result only requires a geometric interpretation of the condition (3.3). Consider a $\mathcal{C}^\infty$ vector field V on $\mathbb{R}^n$:

$$V = \sum_{j=1}^n v_j(x) D_j. \tag{3.50}$$

Then V is *tangent* to H if $Vx_1 = 0$ on $H = \{x_1 = 0\}$. This just means

$$V = v_1'(x) x_1 D_1 + \sum_{j=2}^n v_j(x) D_j \tag{3.51}$$

with $\mathcal{C}^\infty$coefficients. Thus if V is tangent to H then

$$Vu \in H^s(\mathbb{R}^n) \text{ if } u \in I_c^{(s)}(\mathbb{R}^n, H). \tag{3.52}$$

In fact this means that (3.3) can be written as

$$\begin{array}{c}\text{If } V_1 \dots V_p \text{ are any } \mathcal{C}^\infty \text{ vector fields tangent to } H \\ \text{then } V_1 \dots V_p u \in H^s(\mathbb{R}^n) \quad (\forall\ p).\end{array} \tag{3.53}$$

In this form it is clearly invariant under any diffeomorphism preserving H, since the condition of tangency of a vector field is so invariant. □

To extend this coordinate invariance to the space $I^m(\mathbb{R}^n, H)$ is not quite so simple. Indeed this is one reason for considering the $I^{(s)}(\mathbb{R}^n, H)$ at all. However we do in fact get:

PROPOSITION 3.11. *If $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism such that $F(H) = H$ then for every m*

$$F^* : I^m(\mathbb{R}^n, H) \longrightarrow I^m(\mathbb{R}^n, H). \tag{3.54}$$

PROOF. To prove (3.54) we shall break F up into pieces using a partition of unity. We can suppose that $u \in I_c^m(\mathbb{R}^n, H)$ has its support in a small neighbourhood of some point; which we take to be the origin. Since F maps H into itself it is of the form

$$F(x_1, x') = (x_1 a(x), b(x)). \tag{3.55}$$

Thus the map $F_0 : \mathbb{R}^{n-1} \ni x' \longmapsto b(0, x') \in \mathbb{R}^{n-1}$ is also a diffeomorphism. We extend this to a map of the whole space by defining

$$F_0(x_1, x') = (x_1, b(0, x')). \tag{3.56}$$

Consider the form of $F_0^* u$. By definition u is given by

$$u(x_1, x') = \frac{1}{2\pi} \int e^{ix_1\xi} \tilde{u}(\xi, x') d\xi, \quad \tilde{u} \in S^{m+\frac{n}{4}-\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R}). \tag{3.57}$$

Since the inverse Fourier transform is only in the x_1 variable we get immediately

$$F_0^* u(x_1, x') = \frac{1}{2\pi} \int e^{ix_1\xi} \tilde{u}(\xi, b(0, x')) d\xi. \tag{3.58}$$

Thus to show $F_0^* u \in I^m(\mathbb{R}^n, H)$ we just need to observe that

$$\tilde{u}(\xi, b(0, x')) \in S^{m+\frac{n}{4}-\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R}). \tag{3.59}$$

Applying Leibniz' formula repeatedly we see that

$$D^\alpha_\xi D^\beta_{x'} \tilde{u}(\xi, b(0,x')) = \sum_{\gamma\le\beta} b_{\gamma,\beta}(D^\alpha_\xi D^\gamma_{x'}\tilde{u})(\xi, b(0,x')) \tag{3.60}$$

so the estimates (3.59) follows directly from those for $\tilde{u}$.

Thus we have shown that $F_0^* u \in I^m(\mathbb{R}^n, H)$. Put $F_{(1)} = F_0^{-1} \circ F$; this is again a diffeomorphism. By the choice of F_0, $F_{(1)}$ fixes H point wise:

$$F_{(1)}(x_1, x') = (x_1 a(x), x' + x_1 b(x)),\ a(x) \neq 0. \tag{3.61}$$

The next part of F we factor out is the linear part:

$$F_1(x_1, x') = (x_1 a(0,x'), x'). \tag{3.62}$$

Again starting from the representation (3.57) we have

$$\begin{aligned} F_1^* u(x_1, x') &= \frac{1}{2\pi}\int e^{ix_1 a(0,x')\xi}\tilde{u}(\xi, x')d\xi \\ &= \frac{1}{2\pi}\int e^{ix_1\Xi}\ \tilde{u}(\Xi/\, a(0,x'), x')\frac{d\Xi}{a(0,x')}. \end{aligned} \tag{3.63}$$

Thus we need to check that

$$\tilde{U}(\Xi, x') = \tilde{u}(\Xi/a(0,x'), x')/\, a(0,x') \in S^{m+\frac{n}{4}-\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R}), \tag{3.64}$$

since then $U = F_1^* u \in I^m(\mathbb{R}^n, H)$. The proof of (3.64) is once more a consequence of Leibniz' formula and we shall prove it with a simple inductive argument.

LEMMA 3.6. *If $b(\xi, x') \in S^M(\mathbb{R}^{n-1}_{x'}, \mathbb{R})$ then for any non-vanishing $a \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$, and any $c \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$, $l \in \mathbb{N}$*

$$c(x')\xi^l b(\frac{\xi}{a(x')}, x') \in S^{M+l}(\mathbb{R}^{n-1}, \mathbb{R}). \tag{3.65}$$

PROOF. Since $a \neq 0$ we see immediately that the first symbol estimate holds:

$$\left|c(x')\xi^l b(\xi/a(x'), x')\right| \le C_K(1+|\xi|)^{M+l},\ x' \in K. \tag{3.66}$$

By induction one easily sees that

$$D^{\alpha_1}_\xi D^{\alpha'}_{x'}\left[c(x')\xi^l b(\xi/a(x'), x')\right] = \sum_{-l\le p\le|\alpha'|} \xi^{l-p} b_{l,p,\alpha}(\xi/a(x'), x') \tag{3.67}$$

where $a_{l,p,\alpha} \in S^{M+p-\alpha_1}(\mathbb{R}^{n-1}, \mathbb{R})$. Then the general estimate follows from (3.66). □

Thus we have shown (3.64) and hence that $F_1^* u \in I^m(\mathbb{R}^n, H)$. To complete the proof of Proposition 3.11 consider

$$F_{(2)} = F_1^{-1} \circ F_{(1)} = F_1^{-1} \circ F_0^{-1} \circ F. \tag{3.68}$$

By the choice of F_1, in (3.62), $F_{(2)}$ is of the form

$$F_{(2)}(x_1, x') = (x_1 + x_1^2 e_1(x), x' + x_1 e'(x)). \tag{3.69}$$

Not only is this a diffeomorphism but, at least in a small neighbourhood of H, it is connected to the identity by a smooth 1-parameter family of diffeomorphisms of the same type:

$$\begin{gathered} G_t(x) = (x_1 + t x_1^2 e_1(x), x' + t x_1 e'(x'))\ t \in [0,1] \\ G_0(x) = \mathrm{Id},\ G_1(x) = F_{(2)}(x). \end{gathered} \tag{3.70}$$

To show that $F^*_{(2)}u \in I^m_c(\mathbb{R}^n, H)$, where $u \in I^m_c(\mathbb{R}^n, H)$ has support near 0, we use the homotopy method, writing

$$F^*_{(2)}u - u = \int_0^1 \frac{d}{dt} G^*_t u dt. \tag{3.71}$$

Now

$$\frac{d}{dt} G^*_t u = \frac{d}{dt} u(x_1 + tx_1^2 e_1, x' + tx_1 e') = \left[x_1^2 e_1 \partial_{x_1} + x_1 e' \cdot \partial_{x'}\right] G^*_t u. \tag{3.72}$$

At this point we use Proposition 3.10. Thus,

$$u \in I^m(\mathbb{R}^n, H) \Longrightarrow u \in I^{(-m-\frac{n}{4}-1-\epsilon)}(\mathbb{R}^n, H) \text{ for any } \epsilon > 0. \tag{3.73}$$

We already know the coordinate invariance of the Sobolev-based space, from (3.49), so

$$G^*_t u \in I^{(-m-\frac{n}{4}-1-\epsilon)}(\mathbb{R}^n, H) \qquad \forall\ \epsilon > 0. \tag{3.74}$$

Then applying (3.42) again gives

$$G^*_t u \in I^{m+\gamma}(\mathbb{R}^n, H) \qquad \forall\ \gamma > 0. \tag{3.75}$$

Of course, since I^M increases with M, this does not imply that $G^*_t u \in I^m(\mathbb{R}^m, H)$. However consider again (3.72). Differentiating the representation (3.57) we see that

$$\partial_{x_1} G^*_t u \in I^{m+\gamma+1}(\mathbb{R}^n, H),\ \partial_{x'} G^*_t u, G^*_t u \in I^{m+\gamma}(\mathbb{R}^n, H) \quad \forall \gamma > 0. \tag{3.76}$$

The factors in (3.72) can be ordered so that

$$\frac{d}{dt} G^*_t u = x_1^2 \partial_{x_1}(e_1 G^*_t u) + x_1 \partial_{x'}(e' G^*_t u) - x_1^2 \frac{\partial e_1}{\partial x_1} G^*_t u - x_1 \frac{\partial e'}{\partial x'} G^*_t u. \tag{3.77}$$

The same results, (3.76), apply to $\phi \cdot G^*_t u$ if ϕ is $\mathcal{C}^\infty$. Multiplying (3.57) by x_1 and integrating by parts we find

$$x_1 \cdot I^m(\mathbb{R}^n, H) \subset I^{m-1}(\mathbb{R}^n, H) \text{ since } \widetilde{x_1 u}(\xi, x') = -D_\xi \tilde{u}(\xi, x'). \tag{3.78}$$

Now from (3.75) and (3.77) we finally conclude, by choosing $\gamma < 1$, that

$$\frac{d}{dt} G^*_t u \in I^{m-1+\gamma}(\mathbb{R}^n, H) \subset I^m(\mathbb{R}^n, H). \tag{3.79}$$

Thus integrating as in (3.71) we find

$$F^*_{(2)}u - u \in I^m(\mathbb{R}^n, H). \tag{3.80}$$

This completes the proof of the proposition. □

Notice that as a consequence of Proposition 3.11 we get $G^*_t u \in I^m(\mathbb{R}^n, H)$ in (3.72). So in fact

$$F^*_{(2)}u - u \in I^{m-1}(\mathbb{R}^n, H). \tag{3.81}$$

This is important in the treatment of the coordinate invariance of the symbol of a conormal distribution.

Although we have assumed that F is a global diffeomorphism on $\mathbb{R}^n$ in Proposition 3.11 this is by no means necessary. Indeed the proof extends trivially if we simply assume that $F : W \longrightarrow \mathbb{R}^n$ is a diffeomorphism onto its range, $F(W)$, and then

(3.82)

$$F(W \cap H) \subset H, u \in I^m(\mathbb{R}^n, H) \text{ with } \operatorname{supp}(u) \subset W \Longrightarrow F^*u \in I^m(\mathbb{R}^n, H).$$

The most immediate consequence of Proposition 3.11, or rather (3.82), is that we can use it to define the space $I^m(\mathbb{R}^n, H)$ for any hypersurface $H \subset \mathbb{R}^n$ rather than just the model; we now denote $H' = \{x_1 = 0\}$. Recall that a subset $H \subset \mathbb{R}^n$ is an *embedded hypersurface* if for each point $\bar{x} \in H$ there is a $\mathcal{C}^\infty$ map defined in a neighbourhood W of $0 \in \mathbb{R}^n$ which is a diffeomorphism onto its range and satisfies

$$F(0) = \bar{x},\ F(W \cap H') = F(W) \cap H. \tag{3.83}$$

We call such a map a local parametrization of H (near $\bar{x}$).

Definition 3.4. If $H \subset \mathbb{R}^n$ is an embedded hypersurface then we define $I^m(\mathbb{R}^n, H) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ to consist of those distributions satisfying

$$\operatorname{singsupp}(u) \subset H \tag{3.84}$$

and

$$\begin{gathered}\text{If } F : W \longrightarrow \mathbb{R}^n \text{ is a local parametrization of } H \\ \text{then } F^*(\phi u) \in I^m(\mathbb{R}^n, H')\ \forall\ \phi \in \mathcal{C}_c^\infty(F(W)).\end{gathered} \tag{3.85}$$

The first condition, (3.84), guarantees that any element of $I^m(\mathbb{R}^n, H)$ is $\mathcal{C}^\infty$ near the boundary of H, since the singular support is always closed. Of course it also means that these conormal distributions are $\mathcal{C}^\infty$ away from H, which would not automatically follow since (3.85) does not restrict u far away from H. Although $F^*(\phi u)$ is required to be conormal with respect to the model surface H' for every local parametrization of H this is not necessary; rather it suffices to cover H by local parametrizations and then use Proposition 3.11. Each $I^m(\mathbb{R}^n, H)$ is a linear space and the properties of the model case can be transferred to them. First we observe the basic transformation property:

Lemma 3.7. *If $G : U_1 \longrightarrow U_2$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ then for any embedded hypersurface H*

$$G^* : I^m(\mathbb{R}^n, H) \cap \mathcal{C}_c^{-\infty}(U_2) \longrightarrow I^m(\mathbb{R}^n, G^{-1}(H) \cap U_1). \tag{3.86}$$

Proof. By definition any $u \in I^m(\mathbb{R}^n, H)$ can be written as a sum

$$u = u_0 + \sum_{j=1}^{N} F_j^* u_j \tag{3.87}$$

where u_0 is $\mathcal{C}^\infty$, each $F_j : U_j' \longrightarrow \mathbb{R}^n$ is a diffeomorphism onto an open set such that $F^{-1}(H') \subset H$ and $u_j \in I^m(\mathbb{R}^n, H') \cap \mathcal{C}_c^{-\infty}(F_j(U_j'))$. If u has compact support in U_2 then it can be assumed that $U_j' \subset U_2$. Then $G_j = G \circ F_j$ is a diffeomorphism of $U_j'' = G_j^{-1}(U_j')$ onto $f(U_j')$ which maps $G^{-1}(H) \cap U_j''$ onto $H' \cap f(U_j')$; it follows from the representation

$$G^*u = G^*u_0 + \sum_{j=1}^{N} G_j^* u_j \tag{3.88}$$

that $G^*u \in I^m(\mathbb{R}^n, G^{-1}(H)$. □

The order properties are also transferred, so for example

$$I^m(\mathbb{R}^n, H) \subset I^{m'}(\mathbb{R}^n, H) \ \forall \ m \le m'. \tag{3.89}$$

Furthermore if P is any differential operator of order m with $\mathcal{C}^\infty$ coefficients, as in (3.23), then

$$P : I^{m'}(\mathbb{R}^n, H) \longrightarrow I^{m'+m}(\mathbb{R}^n, H). \tag{3.90}$$

It suffices to prove this in case $H = H'$. We have already noted how differentiation acts on these spaces in (3.45) and also, in (3.78), that multiplication by x_1 decreases orders by 1. Thus it suffices to show that $I^m(\mathbb{R}^n, H)$ is a $\mathcal{C}^\infty$-module. Directly from the definition in (3.40) multiplication by a $\mathcal{C}^\infty$ function of x' alone preserves the space. For a general $\mathcal{C}^\infty$ function ϕ writing

$$\phi(x) = \phi(0, x') + \psi(x)x_1, \ \psi \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{3.91}$$

shows that it is enough to check that multiplication by a $\mathcal{C}^\infty$ function increases the order by at most 1. This however follows from the sandwiching relations (3.42) and the fact that the Sobolev-based spaces $I^{(s)}(\mathbb{R}^n, H)$ are $\mathcal{C}^\infty$-modules. Thus we have proved (3.90).

For the model hypersurface recall that a differential operator of order k is characteristic with respect to H' (or H' is characteristic for P) if the coefficient of $D^k_{x_1}$ vanishes at H', i.e. is of the form $x_1 a$ for some $\mathcal{C}^\infty$ function a. Thus

$$\begin{gathered} P \text{ of order } m \text{ is characteristic with respect to } H \\ \Longrightarrow P : I^{m'}(\mathbb{R}^n, H) \longrightarrow I^{m'+m-1}(\mathbb{R}^n, H). \end{gathered} \tag{3.92}$$

Again we have only proved this for $H = H'$, but this suffices to prove the general case since the condition that H be characteristic is coordinate independent.

As we shall see below (3.92) lies at the heart of the construction of plane waves in Chapter 1 and similar constructions below. Althoough we now have enough information on conormal distributions to proceed with the construction of solution to the wave equation we will further investigate the relationship between symbols and conormal distributions.

Conceptually, it is important to understand that symbol spaces and conormal distributions, whilst related by the Fourier transform, are really the same types of distributions. Consider the 'inversion' on $\mathbb{R}$:

$$q : (-1, 1) \ni x \longmapsto \tan \frac{\pi x}{2} \in \mathbb{R}. \tag{3.93}$$

This is certainly an isomorphism being essentially the inverse of stereographic projection. Consider on the closed interval $[-1, 1]$ the following spaces of conormal distribution (on $(-1, 1)$) with respect to the 'hypersurfaces' $x = \pm 1$, defined by weights at the ends

$$\begin{gathered} I^m L^\infty([-1, 1], \{-1\} \cup \{1\}) = \{b \in (1 - |x|^2)^{-m} L^\infty([-1, 1]); \\ [(1 - x^2)^2 D_x]^j b \in (1 - |x|^2)^{-m} L^\infty([-1, 1]) \ \forall \ j \in \mathbb{N}\}. \end{gathered} \tag{3.94}$$

Lemma 3.8. *Pull-back under q in* (3.93) *gives an isomorphism*

$$q^* : S^m(\mathbb{R}) \longleftrightarrow I^m L^\infty([-1, 1]) \ \forall \ m \in \mathbb{R}. \tag{3.95}$$

PROOF. The function $(q^{-1})^*(1-x^2)^{-1}$ is smooth, non-vanishing and of linear growth at infinity in $\mathbb{R}$. Thus

$$|D_\xi^k a(\xi)| \in C_k(1+|\xi|)^{m-k} \ \forall\ k \Longleftrightarrow b = q^* a \in I^m L^\infty([-1,1]) \tag{3.96}$$

as claimed. □

More generally if we define in a similar way

$$\begin{gathered} I^m L^\infty(\mathbb{R}^p \times [-1,1], \mathbb{R}^p \times (\{-1\} \cup \{1\})) \\ = \{b \in (1-|x|^2)^{-m} L^\infty_{\text{loc}}(\mathbb{R}^p \times [-1,1]); \\ \sup_{|y|\le R, x\in(-1,1)} (1-|x|)^m |D_y^\alpha [(1-x^2)^2 D_x]^j b(y,x)| < \infty \\ \forall\ R > 0, \alpha \in \mathbb{N}^p, j \in \mathbb{N}\} \end{gathered} \tag{3.97}$$

we see that

$$q^* : S^m(\mathbb{R}^p; \mathbb{R}) \longleftrightarrow I^m L^\infty(\mathbb{R}^p \times [-1,1], \mathbb{R} \times (\{-1\} \cup \{1\})). \tag{3.98}$$

Thus symbols are just functions which are 'conormal' at infinity, with respect to an inversion.

A further example of this isomorphism allows us to define the symbol spaces on Euclidean space by making a radial inversion. Recall the isomorphism (2.52) between the n-ball and $\mathbb{R}^n$. On the ball we can define spaces similar to (3.98):

$$\begin{gathered} u \in I^m L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1}) \Longleftrightarrow \\ u \in \mathcal{C}^\infty(\overset{\circ}{\to} \mathbb{B}^n) \text{ and } (1-r^2)^m P u \text{ is bounded for all} \\ \mathcal{C}^\infty \text{ differential operators } P \text{ of order } k \text{ such that} \\ (1-r^2)^{-k} P (1-r^2) \in \mathcal{C}^\infty(\mathbb{B}^n). \end{gathered} \tag{3.99}$$

Since this is locally, near the boundary of $\mathbb{B}^n$ the same space as in (3.97) for $p = n-1$ we know how to topologize it in such a way that $\mathcal{C}_c^\infty(\overset{\circ}{\to} \mathbb{B}^n)$ is dense in $I^m L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1})$ in the topology of $I^{m'} L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1})$ for any $m' > m$. In particular if we consider the space

$$I^{m+} L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1}) = \bigcap_{m' > m} I^{m'} L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1}) \tag{3.100}$$

with the topology given by all the seminorms of these spaces then

$$\mathcal{C}_c^\infty(\overset{\circ}{\to} \mathbb{B}^n) \text{ is dense in } I^{m+} L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1}) \ \forall\ m \in \mathbb{R}. \tag{3.101}$$

Then, by analogy with (3.98) we define:

$$\begin{gathered} S^m(\mathbb{R}^n) = Q^*[I^m L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1})], \\ S^{m+}(\mathbb{R}^n) = Q^*[I^{m+} L^\infty(\mathbb{B}^n; \mathbb{S}^{n-1})] \ \forall\ m \in \mathbb{R} \end{gathered} \tag{3.102}$$

and we conclude that

$$\mathcal{C}_c^\infty(\mathbb{R}^n) \text{ is dense in } S^{m+}(\mathbb{R}^n) \ \forall\ m. \tag{3.103}$$

The estimates on an element of $S^m(\mathbb{R}^n)$ are more conventionally written

$$a \in S^m(\mathbb{R}^n) \Longleftrightarrow |D_\xi^\alpha a(\xi)| \le C_\alpha (1+|\xi|)^{m-|\alpha|} \ \forall\ \alpha \in \mathbb{N}^n. \tag{3.104}$$

EXERCISE 3.4. Check that (3.104) is indeed equivalent to (3.102).

Amongst the basic properties of these symbol spaces are

$$D_\xi^\beta : S^m(\mathbb{R}^n) \longrightarrow S^{m-|\beta|}(\mathbb{R}^n) \ \forall \ \beta \in \mathbb{N}^n. \tag{3.105}$$

We shall make use of these spaces to analyse the continuity properties of the Radon transform. First recall Lemma 2.3. Since

$$\mathcal{C}_c^\infty(\mathbb{B}^n) \subset I^{-1}L^\infty(\mathbb{B}^n;\mathbb{S}^{n-1}) \tag{3.106}$$

it follows that

$$R^t : \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow S^{-1}(\mathbb{R}^n). \tag{3.107}$$

Lemma 3.9. *For any* $k \in \mathbb{N}$

$$R^t \cdot D_s^k : \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow S^{-1-k}(\mathbb{R}^n) \subset S^{-1-k+}(\mathbb{R}^n). \tag{3.108}$$

Proof. If $k = 2\ell$ for an integer ℓ then

$$R^t \cdot D_s^{2\ell} = \Delta^\ell R^t. \tag{3.109}$$

In this case (3.108) follows from (3.105). If k is odd, $k = 2\ell+1$ then in place of (3.109) we get

$$R^t \cdot D_s^{2\ell+1} = \Delta^\ell R^t D_s. \tag{3.110}$$

It is therefore enough to show that

$$R^t \cdot D_s : \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow S^{-2}(\mathbb{R}^n). \tag{3.111}$$

This in fact follows from (2.59) which identifies the normal derivative of $R^t u$ at the boundary of the ball. Clearly if $u = D_s v$ this vanishes identically, giving (3.111) and proving the lemma. □

Consider the transpose of this operator with respect to Lebesgue measures. As a direct consequence of (3.108) we find that

$$D_s^k \cdot R : (S^{-1-k+}(\mathbb{R}^n))' \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}) \tag{3.112}$$

is a continuous extension from $\mathcal{C}_c^\infty(\mathbb{R}^n)$, where the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in the symbol space $S^{-1-k+}(\mathbb{R}^n)$ ensures that the dual space in (3.112) is a space of (tempered) distributions and that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in it (in the weak topology.) We do not really need to identify this dual space, but we do note that

$$S^p(\mathbb{R}^n) \hookrightarrow \left[S^{m+}(\mathbb{R}^n)\right]' \text{ if } m+p < -n. \tag{3.113}$$

The pairing here just comes from integration and the inclusions

$$S^{m+}(\mathbb{R}^n)\cdot S^p(\mathbb{R}^n) \subset S^{(m+p)+}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \text{ provided } m+p<-n. \tag{3.114}$$

Thus from (3.112) we conclude that

$$D_s^k \cdot R : S^m(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}) \text{ if } m < k+1. \tag{3.115}$$

One immediate consequence of (3.108) is that

$$R^t \cdot D_s^{\frac{n-1}{2}} : \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow S^{-1-\frac{n-1}{2}}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n). \tag{3.116}$$

Moreover if we impose the natural symmetry condition for $n \ge 3$ odd

$$g(-s,-\omega) = (-1)^{\frac{n-1}{2}} g(s,\omega) \tag{3.117}$$

then

$$R^t \cdot D_s^{\frac{n-1}{2}} g = 0, \text{ where } g \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}) \text{ satisfies (3.117)} \Longrightarrow g = 0. \tag{3.118}$$

Indeed setting $h(\rho,\omega) = \mathcal{F}_s g(\rho,\omega)$ we find that

$$(3.119)\quad R^t D_s^{\frac{n-1}{2}} g(x) = R^t \mathcal{F}_s^{-1}[\rho^{\frac{n-1}{2}} h(\rho,\omega)] = \frac{1}{\pi}\int_{\mathbb{R}^n} e^{ix\cdot(\rho\omega)}[|\rho|^{-\frac{n-1}{2}} h(\rho,\omega)] d(\rho\omega)$$

where (3.117) has been used. Since this is the Fourier transform of a square-integrable function on $\mathbb{R}^n$, $R^t D_s^{\frac{n-1}{2}} g = 0$ implies $h = 0$ and hence gives (3.118).

Lemma 3.10. *If $n \geq 3$ is odd the operator $D_s^{\frac{n-1}{2}} \cdot R$ extends by continuity to an isometric isomorphism*

$$(3.120)\quad D_s^{\frac{n-1}{2}} \cdot R : L^2(\mathbb{R}^n) \longrightarrow \{k \in L^2(\mathbb{R}\times\mathbb{S}^{n-1}); g(-s,-\omega) = (-1)^{\frac{n-1}{2}} g(s,\omega)\}$$

and $R^t \cdot D_s^{\frac{n-1}{2}}$ extends by continuity to be its inverse.

Proof. The continuity of $D_s^{\frac{n-1}{2}} \cdot R$ in (3.120) follows from the Plancherel formula (2.30). Moreover from Theorem 2.1 $R^t \cdot D_s^{\frac{n-1}{2}}$ is a left inverse of this operator. The formula (3.119) extends by continuity and shows it to be injective on the space on the right in (3.120). This proves the lemma. □

Remarks: (1) Supports in (3.30), later $u \in H^s$. (2) Define $\delta(x)$ and $H(x)$ after (3.1)? (3) Add a little more detail about the factorization of F following (3.55). (4) Is (3.69) obvious enough? (4) Is the notation in (3.94) defined? (5) More detail for (3.107). (6) Explain duality in (3.112). (6) The proof of Proposition 3.9 should be improved a bit. (7) More detail in the proof of Proposition 3.10.

CHAPTER 4

Parametrix for the Cauchy problem

Using the inversion formula for the Radon transform and the construction of 'plane wave solutions' above we shall construct a parametrix, i.e. solution operator modulo $\mathcal{C}^\infty$ errors, for the Cauchy problem for P_V.

First recall the inversion formula for the Radon transform

$$\mathrm{Id} = \frac{1}{2(2\pi)^{n-1}} R^t \left|D_s\right|^{n-1} R \quad \text{on } \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{4.1}$$

Written out in terms of Schwartz kernels this is the identity:

$$\delta(x-x') = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \left|D\right|^{n-1} \delta\left((x-x')\cdot\omega\right) d\omega. \tag{4.2}$$

Here $\delta \in \mathcal{C}_c^{-\infty}(\mathbb{R})$ and D is differentiation on the real line. If n is odd then (4.2) can be written:

$$\delta(x-x') = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} (D^{n-1}\delta)\left((x-x')\cdot\omega\right) d\omega. \tag{4.3}$$

This decomposes the Dirac delta distribution at the point x' in terms of derivatives of delta distributions across the hyperplanes through that point.

Next recall the approximate solutions to the wave equation with potential which we have already found. Namely there are three $\mathcal{C}^\infty$ functions

$$u_\pm, h \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{S}^{n-1}) \tag{4.4}$$

such that

$$\begin{gathered} u(t,x;s,\omega) = H(t+s-x\cdot\omega)u_+(t,x;s,\omega) \\ +H(t-s+x\cdot\omega)u_-(t,x;s,\omega) + h(t,x;s,\omega) \end{gathered} \tag{4.5}$$

satisfies the conditions:

$$\begin{gathered} P_V u = (D_t^2 - \Delta - V)u = f \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{S}^{n-1}) \\ u|_{t=0} = 0,\ D_t u|_{t=0} = \delta(s - x\cdot\omega). \end{gathered} \tag{4.6}$$

Thus u is given as the sum of two progressing waves associated to the two hyperplanes which are characteristic for P_V and pass through $\{x\cdot\omega = s\}$ at $t=0$.

Now consider the distribution

$$v(t,x;x',\omega) = D_s^{n-1}u(t,x;x'\cdot\omega,\omega) \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1}). \tag{4.7}$$

This is still well defined since the substitution $s = x'\cdot\omega$ is meaningful in (4.5). From (4.6) we have

$$\begin{gathered} P_V v = f(t,x;x',\omega) \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1}) \\ v|_{t=0} = 0 \text{ and } D_t v|_{t=0} = (D^{n-1}\delta)((x-x')\cdot\omega) \end{gathered} \tag{4.8}$$

since differentiation with respect to s commutes with all the operations. Now set

$$G(t,x,x') = \frac{1}{2(2\pi)^n} \int\limits_{\mathbb{S}^{n-1}} v(t,x;x',\omega)d\omega \tag{4.9}$$

and observe from (4.3), and the fact that ω is a parameter, that

$$\begin{gathered} P_V G = F(t,x,x') \in \mathcal{C}^\infty(\mathbb{R}^{2n+1}) \\ G|_{t=0} = 0 \text{ and } D_t G|_{t=0} = \delta(x-x'). \end{gathered} \tag{4.10}$$

This is the main part of a parametrix for the Cauchy problem. To see this, consider G as the kernel of an operator, which we also devote G (with only slight ambiguity),

$$G\phi(t,x) = \int\limits_{\mathbb{R}^n} G(t,x,x')\phi(x')dx' \quad \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{4.11}$$

This satisfies $P_V G = F$, where F is a smoothing operator, having kernel $F(t,x,x')$. This is 'ignorable' as far as singularities are concerned.

Before proceeding further note that we can assume, or rather arrange, that F satisfies

$$D_t^j F(0,x,x') \equiv 0 \ \forall\ j = 0,\dots. \tag{4.12}$$

To do so we add to G a $\mathcal{C}^\infty$ correction term, G', with Taylor series at $t=0$ determined by

$$\begin{gathered} G'(0,x,x') = D_t G'(0,x,x') \equiv 0 \\ D_t^{j+2} G'(0,x,x') = (\Delta + V) D_t^j G'(0,x,x') - D_t^j F(0,x,x') \ \forall\ j \geq 0. \end{gathered} \tag{4.13}$$

The existence of such a function is guaranteed by Borel's lemma.

Once we have arranged (4.12) notice that if $\psi_0, \psi_1 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then

$$u = G\psi_1 + D_t G\psi_0 \tag{4.14}$$

satisfies the conditions:

$$\begin{gathered} P_V u = F\psi_1 + D_t F\psi_1 \\ u|_{t=0} = \psi_0 \text{ and } D_t u|_{t=0} = \psi_1 \end{gathered} \tag{4.15}$$

where we use the commutativity of P_V and D_t. Since the operator

$$(\psi_0, \psi_1) \longmapsto F\psi_1 + D_t F\psi_0 \tag{4.16}$$

is again a smoothing operator this is the parametrix we seek. We shall, of course, want to examine various regularity properties of (4.14). Before doing this we show how (4.15) leads to a forward parametrix for P_V itself. This will eventually allow us to remove the error terms and solve the Cauchy problem, (4.15), exactly.

The passage from a parametrix for the Cauchy problem to a parametrix for P_V is "DuHamel's Principle." Namely if G satisfies (4.10) then set

$$\widetilde{E}(t,x,t',x') = iH(t-t')G(t-t',x,x') \in \mathcal{C}^{-\infty}(\mathbb{R}^{2(n+1)}). \tag{4.17}$$

Of course we have to make sure that this has a meaning, since in general one cannot multiply distributions. Returning to the definition of v in (4.7), and u in (4.5), the distribution

$$\tilde{v}(t,x;t',x',\omega) = H(t-t')D_s^{n-1}u(t-t',x;x',\omega) \tag{4.18}$$

is meaningful because the introduction of $\tau = t - t'$ and $x = t - t' - (x - x') \cdot \omega$ as (independent) variables reduces it to a sum of products $\phi \cdot H(\tau) \cdot (D_s^k \delta)(\chi)$ with ϕ a $\mathcal{C}^\infty$ function. Thus

$$\widetilde{E}(t, x; t', x') = i \frac{1}{2(2\pi)^n} \int \tilde{v}(t, x; t', x', \omega) d\omega \tag{4.19}$$

is well defined.

LEMMA 4.11. *If* G *satisfies* (4.10), *with* (4.12) *valid and* $\widetilde{E}$ *is given by* (4.17), *then*

$$P_V(i\widetilde{E}) = \delta(t - t')\delta(x - x') + \tilde{F} \text{ with } \tilde{F} \in \mathcal{C}^\infty(\mathbb{R}^{2(n+1)}). \tag{4.20}$$

PROOF. Certainly

$$(\Delta + V)\widetilde{E} = iH(t - t') \left[(\Delta + V)G\right](t - t', x, x'), \tag{4.21}$$

so consider the action of D_t^2. Differentiating by Leibniz' formula gives

$$D_t^2 \widetilde{E} = iD_t^2 H \cdot G + 2D_t H \cdot D_t G + H \cdot D_t^2 G. \tag{4.22}$$

Since $D_t H = -i\delta(t - t')$ and $D_t^2 H = i(D_t \delta)(t - t')$ we find

$$D_t^2 \widetilde{E} = (D\delta)(t - t') \cdot G + 2\delta(t - t') \cdot D_t G + iH \cdot D_t^2 G. \tag{4.23}$$

Using the initial conditions for G in (4.15)

$$P_V \widetilde{E} = \delta(t - t')\delta(x - x') + iHF \tag{4.24}$$

The last term is $\mathcal{C}^\infty$ because of (4.18), so this is just (4.20). □

The identity (4.20), together with the support condition which follows directly from the definition:

$$\operatorname{supp}(\widetilde{E}) \subset \{t - t' \geq 0\} \tag{4.25}$$

means that $\widetilde{E}$ is a forward parametrix for P_V. As an operator

$$\widetilde{E}\phi(t, x) = \int \widetilde{E}(t, x, t', x')\phi(t', x') dt' dx' \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}) \tag{4.26}$$

it satisfies

$$\begin{gathered} P_V \widetilde{E} = Id + \tilde{F} \text{ with } \tilde{F} \text{ smoothing} \\ \operatorname{supp}(\widetilde{E}\phi) \subset \{(t, x); t \geq \inf\{t'; (t', x') \in \operatorname{supp}(\phi)\}. \end{gathered} \tag{4.27}$$

The second condition follows from the fact that $\operatorname{supp}(\widetilde{E}) \subset \{t \geq t'\}$ so the 'integral' in (4.26) is limited to $t \geq T$, where T is the infimum of t on the support of ϕ.

It follows from (4.21), or directly from (4.24), that $\tilde{F}$ has the same support conditions, in addition to having a $\mathcal{C}^\infty$ kernel. This makes it a 'Volterra' operator, the analogue of an upper triangular matrix. With a little more pruning of $\widetilde{E}$ we will find that as an operator $(Id + \tilde{F})$ is invertible with inverse given by $Id + R$, with R again a Volterra operator.

To get such invertibility we need to arrange that the support of $\tilde{F}$ meets any set $\{(t, x, t', x'); t \leq t' + C, |x'| \leq C\}$, in a compact set, i.e. is bounded in x if $t - t'$ is bounded above and x' is bounded. To ensure this we will reduce the support of $\widetilde{E}$. We can do so freely provided we do not change the singularities of $\widetilde{E}$, i.e. provided we remove only a $\mathcal{C}^\infty$ term. Thus we wish to locate the singular support of $\widetilde{E}$. We shall examine a simplified form of this question in a rather *ad hoc* way, but then subsequently consider a general result which actually implies the particular conclusion:

LEMMA 4.12. *For* $\widetilde{E}$ *given by* (4.25)

$$\operatorname{singsupp}(\widetilde{E}) \subset \{|t-t'| = |x-x'|\}. \tag{4.28}$$

PROOF. To prove that the singular support of E' is contained in the surface of the 'light cone' we show first that it is $\mathcal{C}^\infty$ inside the cone, then consider the more subtle question of its smoothness outside the cone. Inside the cone $\widetilde{E} = iG$ or 0, as $t > t'$ on $t < t'$. Thus we consider the singular support of G given by (4.9). We proceed to show

$$\operatorname{singsupp}(G) \subset \{|t| = |x-x'|\}. \tag{4.29}$$

Now from (4.7)

$$\operatorname{singsupp}(v) \subset \{t = \pm(x-x')\cdot\omega\}. \tag{4.30}$$

Thus

$$\operatorname{singsupp}(v) \cap [\{t > |x-x'|\} \cup \{t < -|x-x'|\}] = \emptyset. \tag{4.31}$$

Since G is given by (4.9) we certainly have

$$G \text{ is } \mathcal{C}^\infty \text{ in } t < -|x-x'| \text{ and } t > |x-x'|. \tag{4.32}$$

Of course this argument does not work outside the cone, since at every point there v is singular for some value of ω. The secret is that these singularities are erased by the integration over ω in (4.9) because we always integrate across the singular surface. It is this phenomenon which we proceed to examine in detail. □

Remarks: (1) Explain what DuHamel's Principle is? (2) Check the notation, especially G and E and their tilded versions.

CHAPTER 5

Operations on conormal distributions

The two important operations we wish to consider for conormal distributions are pull-back and push-forward. In this chapter we consider the pull-back operation on conormal distributions and some elementary results on push-forward; the discussion is continued in Chapter 7. Initially we shall only consider these for special maps. It turns out that more general cases follow from these particular ones.

First consider pull-back. If $F : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is any $\mathcal{C}^\infty$ map then

$$\begin{gathered} F^* : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^k), \\ F^*(f) = f \circ F. \end{gathered} \tag{5.1}$$

In general this does *not* extend to all distributions. For example if F is a constant map, with image a point $p \in \mathbb{R}^n$, then $F^*\phi = \phi(p)$ for all $\phi \in \mathcal{C}^\infty(\mathbb{R}^k)$ and this cannot extend continuously to distributions. We therefore look for conditions on F such that F^* extends to conormal distributions associated to a given hypersurface H. To start with suppose that F is the embedding of $\mathbb{R}^k$ as the first k factors in $\mathbb{R}^n$

$$F(y_1, \dots, y_k) = (y_1, \dots, y_k, 0, \dots, 0) \quad (k \le n). \tag{5.2}$$

Suppose that $H \subset \mathbb{R}^n$ is a hypersurface, when can we define F^*u, for each $u \in I^m(\mathbb{R}^n, H)$ and show that it is conormal as well? The case we consider is pictured in Figure 1, where H and $F(\mathbb{R}^k)$ meet transversally, in contrast to Figure 2 where they are tangent.

Definition 5.5. The map (5.2) is transversal to the hypersurface $H \subset \mathbb{R}^n$ (written $F \pitchfork H$ or $H \pitchfork F$) if at each point $\bar{x} = F(\bar{y}) \in F(\mathbb{R}^k) \cap H$, $F_*(T_{\bar{y}}\mathbb{R}^n)$ is *not* contained in $T_{\bar{x}}H$ i.e. some linear combination

$$\sum_{j=1}^k a_j F_*(\partial_j) \text{ is } \textit{not} \text{ tangent to } H \text{ at } \bar{x}. \tag{5.3}$$

One case in which (5.3) certainly holds is

$$F(\mathbb{R}^k) \cap H = \emptyset \Longrightarrow F \text{ is transversal to } H. \tag{5.4}$$

units ¡.5pt,.5pt¿ x from -250 to 250, y from -200 to 200 .5pt from -200 0 to 200 0 -180 150 180 -150 / H [l] at 225 0 $F(\mathbb{R}^k)$ [l]

Figure 1. Transversal intersection

units ¡.5pt,.5pt¿ x from -250 to 250, y from -50 to 200 .5pt from -200 0 to 200 0 -180 160 -90 40 0 0 90 40 180 160 / H [l] at 225 0

Figure 2. Non-transversal intersection

Of course this is not very interesting since then

$$F^*\left(I^m(\mathbb{R}^n, H)\right) \subset \mathcal{C}^\infty(\mathbb{R}^k) \tag{5.5}$$

and there are no singularities.

A useful way to restate Definition 5.5 is to let $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a defining function for H,

$$\text{i.e. } H = \{h = 0\},\ dh \neq 0 \text{ on } H. \tag{5.6}$$

Then

$$F \pitchfork H \Longleftrightarrow d(F^*h)(\bar{y}) \neq 0 \text{ if } F(\bar{y}) \in H. \tag{5.7}$$

Indeed the non-vanishing of the differential means that $v \cdot F^*h \neq 0$ for $v = \sum_{j=1}^{k} a_j \partial_j$ a tangent vector to $\mathbb{R}^k$ at $\bar{y}$. Since $v \cdot F^*h = F_*v \cdot h$ where F_*v is a tangent vector at $\bar{x} = F(\bar{y})$ as in (5.3), the equivalence (5.7) follows. From this we conclude

$$\begin{gathered} F \pitchfork H \Longrightarrow F^{-1}(H) = \{y \in \mathbb{R}^k; F(y) \in H\} \\ \text{is an embedded hypersurface in } \mathbb{R}^k. \end{gathered} \tag{5.8}$$

The function F^*h is a defining function for $F^{-1}(H)$. Now we can state our basic result for pull-back.

Proposition 5.12. *If the map* (5.2) *is transversal to a hypersurface* $H \subset \mathbb{R}^n$ *then*

$$F^* : I^m(\mathbb{R}^n, H) \longrightarrow I^M(\mathbb{R}^k, F^{-1}(H)),\ M = m + \frac{n-k}{4}. \tag{5.9}$$

Proof. The definition of the space of conormal distributions in Definition 3.4 is local. Suppose that $u \in I^m(\mathbb{R}^n, H)$. Since F maps $\mathbb{R}^k \setminus F^{-1}(H)$ into $\mathbb{R}^n \setminus H$, F^*u is defined and smooth in the complement of $F^{-1}(H)$. Using a partition of unity, we can therefore suppose that $u \in I^m(\mathbb{R}^n, H)$ has support near some point $\bar{x} \in H$. Let us introduce local coordinates $x'_1, \dots, x'_n$ near $\bar{x}$ in terms of which $H = \{x'_1 = 0\}$. Then

$$u(x') = \frac{1}{2\pi} \int e^{ix'_1\xi} a(\xi, x'') d\xi,\ x'' = (x_2, \dots, x_n), \tag{5.10}$$

with a a symbol of order $m + \frac{n}{4} - \frac{1}{2}$.

The transversality assumption means that $dF^*x'_1 \neq 0$ at $\bar{y}$ with $F(\bar{y}) = \bar{x}$. Near such a point we can introduce $y'_1 = F^*x'_1$ and $y'_j = F^*x'_{J(j)}$ for $j = 2, \dots, k$, with $1 \in J(j) \leq n$, as coordinates. We can further relabel the coordinates x'_j, for $j \geq 2$, so that $y'_j = F^*x'_j$. Thus in these new local coordinates in the range and domain the map is again of the form (5.2). Moreover from (5.2)

$$F^*u(y') = \frac{1}{2\pi} \int e^{iy'_1\xi} a(\xi, y', 0) d\xi. \tag{5.11}$$

Since the order of a is $m + \frac{n}{4} - \frac{1}{2} = m + \frac{n-k}{4} + \frac{k}{4} - \frac{1}{2}$, F^*u is, according to our strange-looking order convention, of order $M = m + \frac{n-k}{4}$ as claimed. □

One or two points about this proof really require further comment. First one should ask in what sense (5.11) is to hold; it defines the left side, so the proposition should say *there exists* a map (5.9). Then the question of its uniqueness arises. The secret here is *continuity* so we stop to consider this subject a little.

A *locally convex* topology on a vector space is given by a family of seminorms. The topology on $\mathcal{C}^\infty(\mathbb{R}^n)$ is uniform convergence of all derivatives on compact sets, i.e. the seminorms are the '$\mathcal{C}^k$ seminorms' on compact sets:

$$\|u\|_{Q,K} = \sup_{x\in K} |Qu(x)|,\ Q \in \operatorname{Diff}^k(X),\ K \subset\subset X. \tag{5.12}$$

Then F^* in (5.1) is a continuous linear map, meaning that for any of the seminorms $\|\cdot\|_{Q,K}$ on $\mathbb{R}^n$ there is a finite set of seminorms, $\|\cdot\|_{R_i,L_i}$, $i = 1, \dots, N$, on $\mathbb{R}^k$ such that

$$\|F^*u\|_{Q,K} \le \sum_{i=1}^N \|u\|_{R_i,L_i}. \tag{5.13}$$

We want to give each $I^m(\mathbb{R}^n, H)$ a topology so that (5.9) becomes continuous too. The topology should be one in which the space is complete. We take first of all the seminorms given by the $\mathcal{C}^k$ seminorms on compact sets not meeting the hypersurface H. Equivalently this means the seminorms

$$\|\phi u\|_{Q,K} \quad \forall\ Q, K \text{ and } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ with } \operatorname{supp}(\phi) \cap H = \emptyset. \tag{5.14}$$

For the seminorms near H we use the local representation (5.10). Thus if $f : \Omega \longrightarrow \mathbb{R}^n$ is a local coordinate system in which $f(H) \subset \{x_1 = 0\}$ we know that if $\phi \in \mathcal{C}_c^\infty(\Omega)$ there is a unique $v \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ with $\operatorname{supp}(v) \subset f(\Omega)$ and

$$\phi u = f^* v. \tag{5.15}$$

By definition $v \in I^m(\mathbb{R}^n, \{x_1 = 0\})$. This in turn just means that

$$\sup_{(x',\xi)} \left| D_\xi^l D_{x'}^{\alpha'} \tilde{v}(\xi, x') \right| (1 + |\xi|)^{-m-\frac{n}{4}+\frac{1}{2}+l} < \infty \quad \forall\ l, \alpha'. \tag{5.16}$$

Here the conditions that $\tilde{v}$ be a symbol have been written out as the finiteness of a family of seminorms. Thus

Definition 5.6. The topology on $I^m(\mathbb{R}^n, H)$ is given by the seminorms (5.14) and the seminorms (5.16) as f ranges over local coordinate systems flattening H to $\{x_1 = 0\}$ and ϕ has compact support in the coordinate patch.

Lemma 5.13. *With this topology $I^m(\mathbb{R}^m, H)$ is a complete locally convex topological vector space, moreover $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $I^m(\mathbb{R}^n, H)$ in the topology of $I^{m'}(\mathbb{R}^n, H)$ for any $m' > m$.*

Proof. We leave the completeness as an exercise. The main part of the proof of the density is just to show that

$$\text{if } a \in S^m(\mathbb{R}^{k-1}, \mathbb{R}) \text{ and } \mu(\xi) = 1 \text{ in } |\xi| < 1 \text{ then } a_n = \mu(\xi/n)a \longrightarrow a \tag{5.17}$$

in the sense that

$$\begin{gathered}\sup (1 + |\xi|)^{-m+l-\epsilon} \left| D_\xi^l D_{x'}^{\alpha'} (a - a_n) \right| \longrightarrow 0 \\ \forall\ l, \alpha' \text{ and } \epsilon > 0.\end{gathered} \tag{5.18}$$

The estimates for $l = 0, \alpha' = 0$ are easy, since

$$
\begin{aligned}
&\sup(1+|\xi|)^{-m-\epsilon}(1-\mu(\frac{\xi}{n}))|a| \\
&\le C\sup(1+|\xi|)^{-\epsilon}\left|1-\mu(\frac{\xi}{n})\right| \\
&\le C'\sup(1+n)^{-\epsilon} \longrightarrow 0.
\end{aligned} \tag{5.19}
$$

The general case follows similarly. Thus if $u \in I_c^m(\mathbb{R}^n, \{x_1 = 0\})$ and

$$
u_n = \frac{1}{2\pi}\int e^{ix_1\xi}\tilde{u}(\xi, x')\mu(\frac{\xi}{n})d\xi \tag{5.20}
$$

then it follows that $\phi u_n \longrightarrow u$, in the topology of $I^{m'}(\mathbb{R}^n, \{x_1 = 0\})$ for any $m' > m$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ near $\operatorname{supp}(u)$. Density in case of a general H then follows by use of a partition of unity. □

Notice that this proof cannot be strengthened to show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $I^m(\mathbb{R}^n, H)$ in its own topology, rather than that of $I^{m'}(\mathbb{R}^n, H)$ for $m' > m$; in fact it is not dense in this stronger sense.

The precise meaning of (5.9) should now be clear. The formula (5.11) gives a map (5.9) which is continuous in the topology just defined. The fact that the map is independent of m (i.e. F^*u is the same if $u \in I^m(\mathbb{R}^n, H)$ is regarded as a element of $I^{m'}(\mathbb{R}^n, H)$ for some $m' > m$) and the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ just shown means that F^* is the unique continuous extension.

Next observe that the restriction to the case $k < n$ in (5.2) is not important. If $k > n$ consider instead the projection:

$$
F(y_1, \dots, y_k) = (y_1, \dots, y_n) \quad n < k, \tag{5.21}
$$

and in case $n = k$ we get the identity map. The proof of (5.9) proceeds unchanged, except that the coordinates y' in $\mathbb{R}^k$ should be taken to be *all* the $F^*x'_j$, plus the y_j for $j > n$. Then (5.11) becomes

$$
F^*u = \frac{1}{2\pi}\int e^{iy_1'\xi}a(\xi, y_1, \dots, y_k)d\xi, \tag{5.22}
$$

and the result follows as before.

This extension of the pull-back operation allows us to make a considerable further generalization to a much wider class of maps, F. The condition of Definition 5.5 makes sense for *any* $\mathcal{C}^\infty$ map

$$
F : \mathbb{R}^k \longrightarrow \mathbb{R}^n. \tag{5.23}
$$

PROPOSITION 5.13. *Suppose F is any $\mathcal{C}^\infty$ map, as in (5.23), which is transversal to an embedded hypersurface $H \subset \mathbb{R}^n$, then $F^{-1}(H)$ is an embedded hypersurface and*

$$
F^* : I^m(\mathbb{R}^n, H) \longrightarrow I^M(\mathbb{R}^k, F^{-1}(H)) \quad M = m + \frac{n-k}{4} \tag{5.24}
$$

is well defined by continuity from (5.1).

PROOF. The trick here is to consider some related maps. First is the graph-map of F :

$$
F_{\text{gr}} : \mathbb{R}^k \longrightarrow \mathbb{R}^k \times \mathbb{R}^n, \; F_{\text{gr}}(y) = (y, F(y)) \tag{5.25}
$$

and then the projection

$$\pi : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n. \tag{5.26}$$

The second of these is of the form (5.21), if the coordinates are suitably relabelled. The first is locally of the form (5.2). Indeed the map

$$G_F : \mathbb{R}^{k+n} \longrightarrow \mathbb{R}^{k+n}, G_F(y,x) = (y, x+F(y)) \tag{5.27}$$

is a diffeomorphism and

$$F_{\text{gr}} = G_F \circ F', \;\; F'(y) = (y,0). \tag{5.28}$$

This decomposes F into a product

$$F = \pi \circ G_F \circ F' \tag{5.29}$$

of three maps. To the outer two maps (5.9) applies and the invariance of the conormal spaces under diffeomorphisms was discussed earlier. Thus the general case of (5.9) follows from

$$(\pi)^* : I^m(\mathbb{R}^n, H) \longrightarrow I^{m-\frac{k}{4}}(\mathbb{R}^{k+n}, \mathbb{R}^k \times H) \tag{5.30}$$

$$(G_F)^* : I^{m-\frac{k}{4}}(\mathbb{R}^{k+n}, \mathbb{R}^k \times H) \longleftrightarrow I^{m-\frac{k}{4}}(\mathbb{R}^{k+4}, G_F^{-1}(\mathbb{R}^k \times H)) \tag{5.31}$$

$$(F')^* : I^{m-\frac{k}{4}}(\mathbb{R}^{k+n}, G_F(\mathbb{R}^k \times H)) \longrightarrow I^{m+\frac{n-k}{4}}(\mathbb{R}^k, F^{-1}(H)). \tag{5.32}$$

The first two are immediate. The third follows if we check that F' is transversal to $G_F^{-1}(\mathbb{R}^k \times H)$ and

$$(F')^{-1}(G_F^{-1}(\mathbb{R}^k \times H)) = F^{-1}(H). \tag{5.33}$$

This however follows from (5.28), so the proposition is proved. □

One simple example of the use of pull-back, as in Proposition 5.13 is to note that for each $\omega \in \mathbb{S}^{n-1}$ the map

$$F_\omega : \mathbb{R} \times \mathbb{R}^n \ni (t,x) \longmapsto t - x \cdot \omega \in \mathbb{R} \tag{5.34}$$

is transversal to the point-hypersurface $\{0\} \subset \mathbb{R}$. Thus the distributions that we have already been freely using (!) can be interpreted this way:

$$\delta(t - x \cdot \omega) = F_\omega^* \delta_0. \tag{5.35}$$

We shall make more substantial use of pull-back later.

Next we turn to consideration of the push-forward operation. This is dual to pull-back. Thus if $F : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is any $\mathcal{C}^\infty$ map and $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^k)$

$$F_*(u)(\phi) = u(F^*\phi) \qquad \forall \; \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.36}$$

It is important to note that the push-forward of a distribution with compact support under any $\mathcal{C}^\infty$ map is well-defined. In fact

$$F_* : \mathcal{C}_c^{-\infty}(\mathbb{R}^k) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n) \tag{5.37}$$

is continuous, since it is the dual to a continuous map. This is in contrast to the pull-back operation. However, as a manifestation of the 'preservation of mathematical difficulty' trouble appears elsewhere. Namely, in general,

$$F_*(\mathcal{C}_c^\infty(\mathbb{R}^k)) \not\subset \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.38}$$

Thus push-forward can introduce singularities. For example if $F : \mathbb{R}^k \longrightarrow \mathbb{R}$ has $F(x) = 0$ for all $x \in \mathbb{R}^k$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ then

$$F_*(\phi) = (\int \phi dx) \cdot \delta_0. \tag{5.39}$$

EXERCISE 5.5. Prove (5.39).

The natural question we ask is when is $F_*(I_c^m(\mathbb{R}^k, H))$, for $H \subset \mathbb{R}^k$ a hypersurface, contained in the space of conormal distributions associated to some hypersurface. We find an answer to this later, for the moment we just investigate the singular support of $F_*(u)$ when u is conormal.

Let F be the projection (5.21), and consider any $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^k)$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^k)$ then

$$F^*\phi(y, \theta) = \phi(y) \tag{5.40}$$

where we write the coordinates $x_1, \ldots, x_k$ in $\mathbb{R}^k$ in the form $y_j = x_j,\ j \leq n$, and $\theta_j = x_{j-n},\ 1 \leq j \leq p = k - n$. Thus

$$F_*u(\phi) = \int u(y, \theta)\phi(y) dy d\theta = \langle \int u(y, \theta') d\theta', \phi \rangle, \tag{5.41}$$

at least formally, i.e.

$$F_*u(y) = \int u(y, \theta) d\theta \quad \text{if } F \text{ is as in (5.21)} \tag{5.42}$$

In this case certainly $u \in \mathcal{C}_c^\infty(\mathbb{R}^k) \Longrightarrow F_*u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since, for this map, it is clear that

$$\operatorname{supp}(F_*u) \subset F(\operatorname{supp}(u)) \tag{5.43}$$

we deduce that

$$\operatorname{singsupp}(F_*u) \subset F(\operatorname{singsupp}(u)) \quad \forall\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^k). \tag{5.44}$$

This simple result can be strengthened significantly in the case of conormal distributions. If $H \subset \mathbb{R}^k$ is a hypersurface (i.e. a smoth submanifold of codimension one) and F is given by (5.21) set

$$C_H = \{x \in H; H \text{ is tangent to the fibre of } F \text{ at } x\}, \tag{5.45}$$

where the fibre of F through x is $\{x'; F(x') = F(x)\}$. This definition can be rewritten in various ways. The most important is in terms of any defining function $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ of H :

$$\bar{x} \in C_H \Longleftrightarrow \bar{x} \in H, \bar{x} = (\bar{y}, \bar{\theta}) \text{ and } d_\theta h = 0 \text{ at } (\bar{y}, \bar{\theta}). \tag{5.46}$$

Thus at a point of C_H any defining function h of H is stationary when restricted to the fibre of F through the point.

PROPOSITION 5.14. *If F is the map (5.21) and $H \subset \mathbb{R}^k$ is a $\mathcal{C}^\infty$ hypersurface then*

$$\operatorname{singsupp}(F_*u) \subset F(C_H) \quad \forall\ u \in I_c^m(\mathbb{R}^k, H). \tag{5.47}$$

PROOF. We want to show that if $\bar{y} \in \mathbb{R}^n$ is such that

$$(\bar{y}, \theta) \notin C_H \ \forall\ \theta \in \mathbb{R}^{k-n} \text{ then for } u \in I_c^m(\mathbb{R}^k, H) \\ F_*(u) \text{ is } \mathcal{C}^\infty \text{ near } \bar{y}. \tag{5.48}$$

This is just saying that $\phi F_*(u) \in \mathcal{C}^\infty(\mathbb{R}^k)$ for some $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^k)$ with $\phi(\bar{y}) \neq 0$. We can assume that the support of u is in a small neighbourhood of $\bar{x} \in F^{-1}(\bar{y})$, and then use a partition of unity to prove the general result. The condition $\bar{x} \notin C_H$, which follows from (5.48) if $\bar{x} \in H$, means that

$$d_\theta h \neq 0 \text{ at } \bar{x}. \tag{5.49}$$

Renumbering the θ variables we can therefore assume that

$$\partial_{\theta_p} h(\bar{y}, \bar{\theta}) \neq 0,\ p = k - n. \tag{5.50}$$

The implicit function theorem shows that locally near $(\bar{y}, \bar{\theta})$

$$h(y, \theta) = g(y, \theta)(\theta_p - \tau(y, \theta')),\ \theta' = \theta_1, \ldots, \theta_{p-1},\ g \neq 0,\ p = k - n \tag{5.51}$$

with g and τ both $\mathcal{C}^\infty$ functions. Thus $\theta_p - \tau(y, \theta')$ is also a defining function for H. The diffeomorphism

$$(y, \theta) \longmapsto (y, \theta', \theta_p + \tau(y, \theta')) \tag{5.52}$$

reduces H to $\{\theta_p = 0\}$. Thus $u \in I_c^m(\mathbb{R}^k, H)$ with support in a small neighbourhood of $(\bar{y}, \bar{\theta})$ can be written

$$u(y, \theta) = \frac{1}{2\pi} \int e^{i(\theta_p - \tau(y, \theta'))\xi} a(y, \theta', \xi) d\xi. \tag{5.53}$$

Then

$$F_* u(y) = \frac{1}{2\pi} \int e^{i(\theta_p - \tau(y, \theta'))\xi} a(y, \theta', \xi) d\xi d\theta \\ = \frac{1}{2\pi} \int e^{i\theta_p' \cdot \xi} a(y, \theta', \xi) d\xi d\theta' d\theta_p' \tag{5.54}$$

where the transformation $\theta_p' = \theta_p - \tau(y, \theta')$ has been made in the variables of integration. From the continuity of the push-forward map we know that

$$u \in I_c^m(\mathbb{R}^k, H) \Longrightarrow F_* u \in H^M(\mathbb{R}^n), \quad M = M(m), \tag{5.55}$$

i.e. $F_* u$ is in a fixed Sobolev space depending only on the order of the symbols. Since

$$D_y^\alpha F_* u = \frac{1}{2\pi} \int e^{i\theta_p' \xi} (D_y^\alpha a)(y, \theta', \xi) d\xi d\theta' d\theta_p' \tag{5.56}$$

corresponds to a symbol of the same order, independent of α, we conclude that $F_* u \in \mathcal{C}^\infty(\mathbb{R}^n)$. This proves the proposition. □

Although Proposition 5.14 applies only to F of the form (5.21) we can again easily extend it. A $\mathcal{C}^\infty$ map (5.23) is said to be a *submersion* if its differential has constant rank. The implicit function theorem shows that by a change of coordinates in both range and domain it can always be brought, locally, to the form (5.21). Since the hypothesis in Proposition 5.14 is stated in a coordinate-independent form we can generalize it as follows.

Corollary 5.2. *If $F : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a submersion and $H \subset \mathbb{R}^k$ is an embedded hypersurface with*

$$C_{H,F} = \{x \in \mathbb{R}^k; \exists\ v \in T_x\mathbb{R}^k, v \in T_xH \ \textit{but}\ F_*(v) \neq 0\} \tag{5.57}$$

then

$$\operatorname{singsupp}(F_*u) \subset F(C_{H,F}) \quad \forall\ u \in I^m_c(\mathbb{R}^k, H). \tag{5.58}$$

Another case in which we can analyse the form of the push-forward is for the map (5.2), where $k \leq n$. Then for any $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^k)$

$$F_*u(\phi) = u(F^*\phi) = \int u(y)\phi(y, 0, \dots, 0)dy. \tag{5.59}$$

In this case

$$F_*(u)(x) = u(x_1, \dots, x_k) \tag{5.60}$$

and we see immediately that

$$F_*(u) \in I^{m+\frac{n-k}{4}}(\mathbb{R}^n, H \times \mathbb{R}^p) \text{ if } u \in I^m_c(\mathbb{R}^k, H) \text{ and } F \text{ is as in (5.2).} \tag{5.61}$$

We shall examine more general push-forward results in Chapter 7.

Remarks: Clarify comment in paragraph after (5.20). Improve sentence surrounding (5.29).

CHAPTER 6

Forward fundamental solution

Next we proceed to the construction of the forward fundamental solution for the perturbed wave operator P_V. The results of Chapter 5 are used to bound the singular support of a parametrix in the course of the construction.

THEOREM 6.2. *If* $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *there is a unique distribution* $E \in \mathcal{C}^{-\infty}(\mathbb{R}^{2n+1})$ *with the properties*

$$\operatorname{supp}(E) \subset \{t \geq 0\} = [0, \infty) \times \mathbb{R}^{2n} \text{ and} \tag{6.1}$$

$$P_V E(t, x, x') = \delta(t)\delta(x - x'); \tag{6.2}$$

this distribution has the two additional properties

$$\operatorname{supp}(E) \subset \{t \geq |x - x'|\} \text{ and} \tag{6.3}$$

$$\operatorname{singsupp}(E) \subset \{t = |x - x'|\}. \tag{6.4}$$

We shall construct E satisfying (6.1) and (6.2) by iteration using the forward parametrix found in Lemma 4.11, after a little surgery. We then use a simple duality argument to conclude the uniqueness of such a forward fundamental solution and hence get the improved support estimate (6.3).

Our pruning of the forward parametrix is based on the bound (4.28), i.e. (6.4), for the singular support of E' defined by (4.25). The non-rigorous argument following Lemma 4.12 is bolstered by Proposition 5.14. Thus in (4.25) we know that G is given by (4.9) where

$$v = v_+ + v_-, \ v_\pm \in I^{\frac{n}{4}-\frac{3}{2}}(\mathbb{R}^{2n+1} \times \mathbb{S}^{n-1}, \{t = \pm(x - x') \cdot \omega\}). \tag{6.5}$$

For either of these two hypersurfaces to be tangent to the ω-fibres in the push-forward in (4.9) requires that $x = x'$ or

$$d_\omega \left[(x - x') \cdot \omega\right] = x - x' - [(x - x') \cdot \omega]\omega = 0. \tag{6.6}$$

Since $t = \pm(x - x') \cdot \omega$ this certainly implies that $|t| = |x - x'|$. Thus the singular support of G in (4.9) is contained in $|t| = |x - x'|$. In (4.12) and (4.13), G has been modified by a $\mathcal{C}^\infty$ term so that its Taylor series vanishes at $t = 0$, at least in $x \neq x'$. When E' is defined by (4.25) it therefore does not acquire any singularities at $t = 0$, away from $x = x'$. This completes the proof of Lemma 4.12.

Now we know that E' in (4.25) is smooth away from the forward light cone $t = |x - x'|$ (since it is zero in $t < 0$), we can reduce its support, without destroying the parametrix property. Ideally we should like to find a parametrix with support in the cone $t \geq |x - x'|$ but we cannot do this directly since we do not know at this stage that E' is smooth up to the cone from the outside (although it is.) Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a cut-off function chosen so that

$$\rho(s) = 1 \text{ if } s > -1, \ \rho(s) = 0 \text{ if } s < -2. \tag{6.7}$$

00 0 ¡4pt¿ [.2,.67] from 0 -180 to 0 180 ¡4pt¿ [.2,.67] from 150 75 to 154 77 -150 -75 150 75 / 0 0 90 180 / 0 0 -90 180 / axes ratio 3:1 360

FIGURE 1. Support of E''

Then consider

$$E''(t,x,x') = \rho(t - |x - x'| - \frac{1}{2})E'(t,x,x'). \tag{6.8}$$

This is well-defined since $\rho(t - |x - x'| - \frac{1}{2})$ is $\mathcal{C}^\infty$ on the support of E', being constant near the singular set $t = |x - x'|$ in $t \geq 0$. Moreover

$$E'' - E' \in \mathcal{C}^\infty(\mathbb{R}^{2n+1}) \text{ and } \operatorname{supp}(E'') \subset \{t \geq 0\} \cap \{t \geq |x - x'| - \frac{3}{2}\}. \tag{6.9}$$

This modified version of the parametrix satisfies

$$\begin{gathered} P_V E'' = \delta(t)\delta(x - x') - R(t,x,x'),\ R \in \mathcal{C}^\infty(\mathbb{R}^{2n+1}), \\ \operatorname{supp}(R) \subset \{t \geq 0\} \cap \{t \geq |x - x'| - 2\}. \end{gathered} \tag{6.10}$$

The remainder term here defines an operator, as usual denoted by R,

$$Rf(t,x) = \int R(t - t', x, x')f(t', x')dt'dx'. \tag{6.11}$$

The support condition on the kernel in (6.10) means that R extends by continuity to some distributions without compact support. The distributions to which it can be applied are as follows:

DEFINITION 6.7. A distribution $u \in \mathcal{C}^{-\infty}(\mathbb{R}^{n+1})$ is said to have *past-compact* support if

$$\begin{gathered} \text{for every } K \subset\subset \mathbb{R}^{n+1}, \\ \{(t', x') \in \operatorname{supp}(u); t' - t \leq -|x - x'| \text{ for some } (t,x) \in K\} \subset\subset \mathbb{R}^{n+1}. \end{gathered} \tag{6.12}$$

Let $\mathcal{C}^{-\infty}_{\text{PC}}([0,\infty) \times \mathbb{R}^n)$ denote the subspace of $\mathcal{C}^{-\infty}(\mathbb{R}^{n+1})$ consisting of those distributions with past-compact support. Let $C^-_{(t',x')}$ be the past light cone of P_V based at (t', x') :

$$C^-_{(t',x')} = \{(t,x) \in \mathbb{R}^{n+1}; t - t' \leq -|x - x'|\}. \tag{6.13}$$

The condition (6.12) is equivalent to

$$C^-_{(a,0)} \cap \operatorname{supp}(u) \subset\subset \mathbb{R}^{n+1}\ \forall\ a \in \mathbb{R} \Longleftrightarrow u \in \mathcal{C}^{-\infty}_{\text{PC}}(\mathbb{R}^{n+1}) \tag{6.14}$$

since for each fixed point (t', x') when a is large enough $C^-_{(t',x')} \subset C^-_{(a,0)}$. If $\mu \in \mathcal{C}^\infty(\mathbb{R})$ has $\mu(s) = 1$ in $s < -\frac{3}{4}$ and $\mu(s) = 0$ in $s > -\frac{1}{4}$ then $\phi(t,x) = \mu(t - 1 + (\frac{1}{4} + |x|^2)^{-\frac{1}{2}})$ satisfies

$$\phi(t,x) = 0 \text{ if } (t,x) \notin C^-_{(1,0)},\ \phi(t,x) = 1 \text{ if } (t,x) \in C_{(0,0)}. \tag{6.15}$$

Set $\phi_a(t,x) = \phi(t - a, x)$ for any $a \in \mathbb{R}$. Then if $u \in \mathcal{C}^{-\infty}_{\text{PC}}(\mathbb{R}^{n+1})$, $\phi_a u$ has compact support. We can topologize $\mathcal{C}^{-\infty}_{\text{PC}}(\mathbb{R}^{n+1})$ as a complete locally convex topological vector space by using the seminorms from $\mathcal{C}^{-\infty}(\mathbb{R}^{n+1})$ together with the seminorms of $\mathcal{C}^{-\infty}_c(\mathbb{R}^{n+1})$ on $\phi_a u$ for each a. The related space of smooth functions, $\mathcal{C}^\infty_{\text{PC}}(\mathbb{R}^{n+1})$, is defined by the same support conditions, with $\mathcal{C}^{-\infty}$ replaced by $\mathcal{C}^\infty$ throughout.

LEMMA 6.14. *The operator, R, defined by* (6.11) *extends to a continuous linear operator*

$$R : \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}). \tag{6.16}$$

PROOF. Suppose $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$. Then, for any $a \in \mathbb{R}$, $\psi R(\phi_a u)$ is well-defined (and $\mathcal{C}^\infty$) if $u \in \mathcal{C}^{-\infty}_{\mathrm{PC}}\mathbb{R}^{n+1})$ since $\phi_a u$ has compact support. In fact

$$\psi R u = \lim_{a\to\infty} \psi R(\phi_a u) \tag{6.17}$$

is actually independent of a for a sufficiently large. To see this just note that the support of the kernel of the operator ψR is contained in the set

$$\{(t,x,t',x'); t - t' > |x - x'| - 2 \text{ for some } (t,x) \in \operatorname{supp}(\psi)\}. \tag{6.18}$$

This is contained in the region where $\phi_a = 1$ if a is large enough. Thus Ru is well-defined by (6.17) if u has past compact support. The same type of argument shows that Ru also has past-compact support. □

Since the proof of Lemma 6.14 only uses the support property of the kernel if follows that the parametrix E'' has the same property, without being regularizing:

$$E'' : \mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}). \tag{6.19}$$

In fact more is true, namely E'' acts on both $\mathcal{C}^\infty$ functions and distributions.

LEMMA 6.15. *The parametrix E'' gives a continuous linear operator*

$$E'' : \mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \tag{6.20}$$

and extends by continuity to

$$E'' : \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}). \tag{6.21}$$

PROOF. Suppose $f \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$. The definition of $E''f$ can be written in terms of

$$\begin{aligned} M_\pm f(t,x,t',\omega) &= \int_{\mathbb{R}^n} v_\pm(t-t',x,x',\omega) f(t',x')dx', \\ Q(t,x,t',\omega) &= \int_{\mathbb{R}^n} h(t-t',x,x',\omega) f(t',x')dx' \end{aligned} \tag{6.22}$$

as

$$E''f(t,x) = \int_{-\infty}^{t} \int_{\mathbb{S}^{n-1}} \left[M_+(t,x,t',\omega) + M_-(t,x,t',\omega) + Q(t,x,t',\omega)\right] d\omega dt. \tag{6.23}$$

Now Q in (6.22) is certainly $\mathcal{C}^\infty$, but so indeed are $M_\pm$. To see this just interpret the x'-integral as push-forward, and apply Proposition 5.14. The x'-fibres are never tangent to the hypersurface $t - t' = (x - x') \cdot \omega$ so $M_\pm$, and hence $E''f$, are $\mathcal{C}^\infty$. Thus we have shown

$$E'' : \mathcal{C}^{\infty}_{c}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{\infty}(\mathbb{R}^{n+1}). \tag{6.24}$$

However (6.20) follows from this and (6.19), because

$$\mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) = \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \cap \mathcal{C}^{\infty}(\mathbb{R}^{n+1}). \tag{6.25}$$

To prove (6.21) we can use a duality argument. The transpose of $(E'')^t$ has kernel $(E'')^t(t,x,x') = E''(-t,x',x)$. It therefore also has the mapping property (6.24). Thus, by duality

$$E'' : \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+1}) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^{n+1}). \tag{6.26}$$

From this (6.21) follows. □

Consider again the identity (6.10). Under the reflection $t \to -t$ the wave operator is preserved, so writing E''_- for the t-convolution operator with kernel $E''(-t,x,x')$ and R_- for $R(-t,x,x')$, we get

$$P_V \cdot E''_- = \operatorname{Id} - R_. \tag{6.27}$$

Taking the transpose of (6.27) and denoting $(E''_-)^t(t,x,x')$ by $\tilde{E}''(t,x,x') = E''(t,x',x)$ and $R^t(t,x,x')$ by $\tilde{R}(t,x,x') = R(t,x',x)$ we find that $\tilde{E}''$ is a *left forward parametrix*

$$\tilde{E}'' \cdot P_V = \operatorname{Id} - \tilde{R}. \tag{6.28}$$

The importance of (6.28) is that it gives regularity results for solutions of the equation. Note that $\tilde{E}''$ also has the mapping properties (6.20) and (6.21) and hence that (6.28) holds as an identity for operators on $\mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1})$. Thus we conclude that

$$P_V u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}),\ u \in \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \Longrightarrow u \in \mathcal{C}^\infty_{\mathrm{PC}}(\mathbb{R}^{n+1}). \tag{6.29}$$

Indeed, applying (6.28)

$$u = \tilde{R}u + \tilde{E}''(P_V u) \tag{6.30}$$

and then (6.21) shows that the second term on the right is $\mathcal{C}^\infty$. Applied to operators this argument gives

Lemma 6.16. *If E_1 and E_2 are two (right) parametrices for P_V which act on $\mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1})$ then $E_1 - E_2$ has a smooth kernel.*

Proof. An operator has a smooth kernel if and only if it maps $\mathcal{C}_c^{-\infty}(\mathbb{R}^{n+1})$ to $\mathcal{C}^\infty(\mathbb{R}^{n+1})$. If $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+1})$ then, by assumption, $u = E_1 f - E_2 f \in \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1})$ and $P_V u$ is $\mathcal{C}^\infty$. Thus (6.29) shows that $E_1 f - E_2 f$ is $\mathcal{C}^\infty$, proving the lemma. □

To make use of Lemma 6.16 we shall recall the construction of the parametrix E''. With some slight modifications this gives other parametrices to which we can usefully apply Lemma 6.16.

Lemma 6.17. *For any $\xi \in \mathbb{R}^n$ with $|\xi| < 1$ there is a right parametrix E''_ξ of P_V satisfying the support conditions*

$$\operatorname{supp}(E''_\xi) \subset \{t \geq (x - x') \cdot \xi\} \cap \{t \geq |x - x'| - 2\}. \tag{6.31}$$

Proof. To construct E''_ξ we retrace the construction of E' and hence $E'' = E''_0$ through (4.25), Lemma 4.11, (4.8) and (4.5) but replacing the initial surface $t = 0$ by $t = x \cdot \xi$. It is straightforward to check that this makes no essential difference, provided $|\xi| < 1$. For example in place of (4.25) we get

$$E''_\xi = iH(t - t' - (x - x') \cdot \xi)\tilde{G}_\xi(t,x,t',x'). \tag{6.32}$$

Again (4.28) holds in view of the argument following (6.5). This proves the lemma. □

We can use this abundance of right forward parametrices and the regularity result Lemma 6.16 to construct a better right parametrix.

Lemma 6.18. *Suppose* $\Xi = \{\xi_1, \dots, \xi_N\} \subset \{\xi \in \mathbb{R}^n; |\xi| < 1\}$ *is any finite set, then there exists a forward, right parametrix* E_Ξ *for* P_V *with*

$$\operatorname{supp}(E_\Xi) \subset L_\Xi = \bigcap_{\xi\in\Xi} \{(t,x,x') \in \mathbb{R}^{n+1}; t \ge (x-x')\cdot\xi\}. \tag{6.33}$$

Proof. We already know this when Ξ has only one element so we proceed by induction over the number of elements of Ξ. Suppose therefore that we know the result for Ξ and wish to add another element ξ, i.e. construct $E_{\Xi'}$ for $\Xi' = \Xi \cup \{\xi\}$. By Lemma 6.16 the difference satisfies

$$E_\Xi - E''_\xi = F \in \mathcal{C}^\infty(\mathbb{R}^{2n+1}) \text{ and } \operatorname{supp}(F) \subset \{t \ge (x-x')\cdot\xi\} \cup L_\Xi. \tag{6.34}$$

Now since the set $t = 0, x = x'$ is in the boundary of the support, F must vanish to infinite order there. By introducing polar coordinates around this set F can be decomposed as a difference

$$\begin{gathered} F = F_\Xi - F_\xi \text{ with } F_\Xi, F_\xi \in \mathcal{C}^\infty(\mathbb{R}^{2n+1}), \\ \operatorname{supp}(F_\Xi) \subset L_\Xi,\ \operatorname{supp}(F_\xi) \subset \{t \ge (x-x')\cdot\xi\}. \end{gathered} \tag{6.35}$$

This of course means that

$$E_{\Xi'} \overset{\text{def}}{\to}= E_\Xi - F_\Xi = E''_\xi - F_\xi \tag{6.36}$$

is a forward parametrix with support in the intersection of the supports, i.e. $L_{\Xi'}$. This proves the lemma. □

By choosing Ξ to be a close enough approximation to the boundary of the ball in $\mathbb{R}^n$ we can certainly arrange that

$$L_\Xi \subset \{t \ge (1-\epsilon)|x-x'|\}, \tag{6.37}$$

for any preassigned $\epsilon > 0$. Choosing such a set Ξ let $E_\epsilon = E_\Xi$ be the corresponding forward parametrix. Thus the support is in a cone only slightly larger than desired for (6.3). Let R_ϵ be the corresponding remainder term in

$$P_V \cdot E_\epsilon = \operatorname{Id} - R_\epsilon. \tag{6.38}$$

The support condition means that all powers of R, as an operator, are defined, since in particular they all act on distributions with past-compact support. Let $R_k(t,x,x')$ be the kernel of R^k as a convolution operator in t.

Lemma 6.19. *For each* $T > 0$ *there is a constant* $C = C(T)$ *such that*

$$|R_k(t,x,x')| \le C^{k+1}\frac{t^k}{k!} \text{ in } t < T. \tag{6.39}$$

Proof. As noted earlier we can assume that all the E_ξ, and hence E_ϵ and the remainder R_ϵ, are functions only of t and $x-x'$ outside some region $|x-x'| < Ct+C$ for C large. This means that R_ϵ satisfies a uniform estimate

$$|R_\epsilon(t,x,x')| \le C'_T \text{ in } t < T. \tag{6.40}$$

The kernels R^k are defined iteratively, with $R^1 = R_\epsilon$ by the integrals

$$R^k(t, x, x') = \int\limits_{|x-x''|<(1-\epsilon)t} \int\limits_0^t R^{k-1}(t - t', x, x'')R_\epsilon(t', x'', x)dt'dx'', \tag{6.41}$$

where the limitation on the domain of integration comes from the fact that $t' \geq 0$ on $\operatorname{supp}(R^{k-1})$ and (6.33). Clearly therefore

$$\operatorname{supp}(R^k) \subset \{(t, x, x'); t \geq (1 - \epsilon)|x - x'|\}. \tag{6.42}$$

More importantly from (6.41) we can get the uniform estimates on the R^k. Inserting (6.39), for $k - 1$, and (6.40) in (6.41) gives the same estimate, (6.39), for R^k (provided C_T is large enough) since the volume of the domain of integration is bounded above. □

Of course the importance of (6.39) is that it shows the Neumann series

$$R'(t, x, x') = \sum_{k=1}^{\infty} R^k(t, x, x') \tag{6.43}$$

to be uniformly exponentially convergent as a series of continuous functions. Thinking in terms of operators we can write

$$R' = R + R^2 + R \cdot R' \cdot R. \tag{6.44}$$

This shows that $R' \in \mathcal{C}^\infty(\mathbb{R}^{2n+1})$. From (6.43) we also have the desired inversion property:

$$(\operatorname{Id} + R) \cdot (\operatorname{Id} - R') = \operatorname{Id}, \tag{6.45}$$

as operators on $\mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1})$. Inserting this in (6.38) gives:

$$E(t, x, x') = E_\epsilon(t, x, x') + \int E_\epsilon(t - t', x, x'')R'(t', x'', x')dx''dt' \tag{6.46}$$

satisfing (6.1) and (6.2), for any $\epsilon > 0$. Here of course R' really depends on ϵ. Moreover,

$$P_V \cdot E = P_V \cdot E_\epsilon - P_V \cdot E_\epsilon \cdot R' = \operatorname{Id} + R_\epsilon - (\operatorname{Id} + R_\epsilon) \cdot R' = \operatorname{Id}, \tag{6.47}$$

which is just (6.2).

Proof. Proof of Theorem 6.2 We have succeeded, according to (6.47) in showing the existence of a forward fundamental solution with support in the cone $t > (1 - \epsilon)|x - x'|$. This is a right inverse for P_V. The argument leading from (6.27) to (6.28), time reversal and the taking of the transpose, shows that it corresponds to a similar left inverse, also acting on distributions with past-compact support. The proof of the regularity result Lemma 6.16 now shows that any two right forward fundamental solutions must be equal. Thus E in (6.46) must indeed be independent of $\epsilon > 0$. This proves the support property (6.3), as well as the uniqueness statement. Finally the estimate (6.4) on the singular support of E follows from (4.28) and the fact that $E - E'$ is $\mathcal{C}^\infty$. Notice also that

$$E(t, x, x') = E(t, x', x) \tag{6.48}$$

so that left and right forward parametrices are equal. Indeed, writing $\tilde{E}$ for the operator with kernel $E(t, x', x)$, (6.48) follows from the usual group identity:

$$\tilde{E} = \tilde{E} \cdot P_V \cdot E = E \tag{6.49}$$

interpreted as an operator equation on $\mathcal{C}^{\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1})$. □

The support property (6.3) of the forward fundamental solution allows us to prove existence and uniqueness for the forcing problem for P_V. Thus, for $T \in \mathbb{R}$ set

$$\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n) = \left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}^{n+1}); \operatorname{supp}(u) \subset [T,\infty) \times \mathbb{R}^n\right\}. \tag{6.50}$$

The topology on $\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n)$ is given by the seminorms on $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ together with the seminorms of $\mathcal{C}^{-\infty}_c(\mathbb{R}^{n+1})$ on $\phi(t,x)u$ where ϕ ranges over those functions in $\mathcal{C}^{\infty}(\mathbb{R}^{n+1})$ with support meeting $[T,\infty) \times \mathbb{R}^n$ in a compact set and u ranges over $\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n)$. It is straightforward to check that $\mathcal{C}^{\infty}_c((T,\infty) \times \mathbb{R}^n)$ is dense in $\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n)$ in this topology.

PROPOSITION 6.15. *For each $T \in \mathbb{R}$ and $V \in \mathcal{C}^{\infty}_c(\mathbb{R}^n)$*

$$P_V : \dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n) \longrightarrow \dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n) \tag{6.51}$$

is an isomorphism.

PROOF. Certainly P_V is a map as in (6.51). Since

$$\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n) \subset \mathcal{C}^{-\infty}_{\mathrm{PC}}(\mathbb{R}^{n+1}) \quad \forall\ T \in \mathbb{R} \tag{6.52}$$

the forward fundamental solution acts continuously on $\dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n)$. Moreover the support property (6.3) implies that

$$E : \dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n) \longrightarrow \dot{\mathcal{C}}^{-\infty}([T,\infty) \times \mathbb{R}^n). \tag{6.53}$$

Thus (6.47) extends by continuity to these spaces. The same applies to the other inversion identity $E \cdot P_V = \mathrm{Id}$ proving the proposition. □

This shows that the forcing problem

$$\begin{gathered} P_V u = f,\ f \in \mathcal{C}^{\infty}(\mathbb{R}^{n+1}), f = 0 \text{ in } t < T \\ \text{has a unique solution with } u = 0 \text{ in } t < T. \end{gathered} \tag{6.54}$$

The time-reversibility of P_V shows that $E_-(t,x,x') = E(t,x,x')$ is the unique backward fundamental solution and so with the obvious notation

$$\begin{gathered} P_V : \dot{\mathcal{C}}^{-\infty}((-\infty,T] \times \mathbb{R}^n) \longrightarrow \dot{\mathcal{C}}^{-\infty}((-\infty,T] \times \mathbb{R}^n) \\ \text{is an isomorphism } \forall\ T \in \mathbb{R}. \end{gathered} \tag{6.55}$$

The forward fundamental solution, E, differs from the parametrix, E', by a smoothing operator. To understand more precisely the structure of the singularity (on the forward light cone) of the kernel, E, we need only analyse the parametrix. The main step in doing this is to discuss with more precision the push-forward of conormal distributions, so that we can examine G in (4.9).

Remark: Have distributions on $\mathbb{S}^{n-1}$ been defined? Check out the G's and E's and their primed and tilded versions. In the statement preceeding (6.40) have we noted earlier ... and if so where?

CHAPTER 7

Symbolic properties of conormal distributions

We shall now introduce a coordinate-independent symbol mapping for conormal distributions associated to a hypersurface and then discuss how the symbol transforms under pull-back, subject to the transversality condition (5.3). Next we discuss a related condition which suffices to ensure that the push-forward of a conormal distribution is conormal and we then again describe the effect on the symbol map.

Recall that we defined the space of conormal distributions of order $m \in \mathbb{R}$ associated to the 'model' hypersurface $H = \{x_1 = 0\}$ by

$$\begin{gathered} u \in I^m_c(\mathbb{R}^n, H) \Longleftrightarrow u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ s.t.} \\ \tilde{u}(\xi, x') = 2^{\frac{n}{2}-1}\pi^{\frac{n}{4}} \int e^{-ix_1\xi} u(x_1, x')dx_1 \in \mathcal{S}^{m+\frac{n}{4}-\frac{1}{2}}(\mathbb{R}^{n-1}; \mathbb{R}). \end{gathered} \tag{7.1}$$

The 'full' symbol of $u \in I^m_c(\mathbb{R}^n, H)$ is this normalized one-dimensional Fourier transform $\tilde{u}$. The presence of the normalization factor is simply so that the map will be more consistent with the symbol map for Lagrangian distributions defined later. Since this factor depends only on the dimension of the space it is not very significant. To get invariance under coordinate transformations we need to check how the symbol $\tilde{u}$ transforms under maps preserving H. To get a simple transformation law it turns out that we have to drop a substantial part of the information contained in $\tilde{u}$. In particular consider the leading part of the symbol

$$\tilde{\sigma}_m(u) = [\tilde{u}] \in \mathcal{S}^{m+\frac{n}{4}-\frac{1}{2}}(\mathbb{R}^{n-1}, \mathbb{R})/S^{m+\frac{n}{4}-\frac{3}{2}}(\mathbb{R}^{n-1}, \mathbb{R}). \tag{7.2}$$

Even then, to get proper invariance for this equivalence class of symbols, we need to add a density factor and set:

$$\sigma_m(u) = \tilde{\sigma}_m(u)|d\xi| \in S^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \otimes \Omega_{\text{fibre}}/S^{m+\frac{n}{4}-\frac{3}{2}}. \tag{7.3}$$

We proceed to explain the notation in (7.3). First observe the convention that we do not, for brevity, repeat the spaces in the denominator, i.e. the quotient in (7.3) is really

$$S^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \otimes \Omega_{\text{fibre}}/S^{m+\frac{n}{4}-\frac{3}{2}}(N^*H) \otimes \Omega_{\text{fibre}}. \tag{7.4}$$

Now the space N^*H is just $\mathbb{R}^{n-1} \times \mathbb{R}$. However we identify it with the conormal bundle of H :

$$N^*_x H = \{dh(x); h \in \mathcal{C}^\infty(\mathbb{R}^n), h = 0 \text{ on } H\}, \quad \forall\ x \in H. \tag{7.5}$$

Of course any function h which vanishes on H is of the form $h = x_1 f$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus the differential of h is just

$$dh = f(0, x')dx_1 = \xi dx_1 \text{ at } x = (0, x') \in H. \tag{7.6}$$

Thus a general point of N^*H is just a pair $(x, \xi dx_1)$, giving the identification

$$N^*H \equiv \mathbb{R}^{n-1} \times \mathbb{R}. \tag{7.7}$$

This identification depends on the coordinate system, so we need to examine how coordinate changes affect $\tilde{u}$. This is done below.

First however we have to define Ω_{fibre}. Suppose V is any real vector space of dimension q. Then consider

$$\Lambda^k V = \{w \in V \otimes \cdots \otimes V; w \text{ is totally antisymmetric}\} \tag{7.8}$$

where antisymmetry is under each exchange of a pair of factors of V. Each of the spaces $\Lambda^k V$ is a vector space. Moreover $\Lambda^q V$ is one-dimensional. Set

$$\Omega V = \{u : \Lambda^q(V^*) \longrightarrow \mathbb{R}; u(s\gamma) = |s| u(\gamma) \quad \forall\, \gamma \in \Lambda^q(V^*)\}. \tag{7.9}$$

Thus ΩV is the space of absolutely homogeneous (*not* linear) functions on $\Lambda^q(V^*)$. Clearly ΩV is itself a one-dimensional vector space. In fact there is a map

$$\Lambda^q V \ni \mu \longmapsto |\mu| \in \Omega \tag{7.10}$$

since each $\mu \in \Lambda^q V$ can be identified with some $\mu' \in (\Lambda^q(V^*))'$. However the map (7.10) is *not* linear.

The space

$$(\Omega_{\text{fibre}}(N^*H))_x = \Omega[(N^*_x H)^*] \quad \forall\, x \in H \tag{7.11}$$

is spanned by $|d\xi|$ at each point $x \in H$. Thus the notation (7.3) means that $\sigma_m(u)$ is to be interpreted as a product (strictly speaking an equivalence class of products)

$$a \in S^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \times b' = b(x)|d\xi| \in \mathcal{C}^\infty(H, \Omega_{\text{fibre}}). \tag{7.12}$$

In this sense the tensor product is over $\mathcal{C}^\infty(H)$, i.e.

$$ab' = \tilde{a}\tilde{b}' \text{ iff } \exists\, g \in \mathcal{C}^\infty(H),\ g \neq 0,\ a = g\tilde{a}, \tilde{b}' = gb'. \tag{7.13}$$

With these preliminaries the definition (7.3) now makes sense. Of course $\sigma_m(u)$ contains precisely the same information about u as does $\tilde{\sigma}_m(u)$. The important property of $\sigma_m(u)$ is that it transforms in a simple way under a change of coordinates. To make sense of this statement we need to note that N^*H and Ω_{fibre} are both *natural* objects. That is they have 'obvious' transformations laws under changes of coordinates i.e. we need to show that any diffeomorphism of $\mathbb{R}^n$ preserving H induces natural maps on N^*H and on Ω_{fibre}.

From the definition, (7.5), of the fibre of N^*H, there is an obvious map

$$\begin{gathered} F^* : N^*H \longrightarrow N^*H \text{ if } F : \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ \text{is a diffeomorphism with } F(H) \subset H. \end{gathered} \tag{7.14}$$

Namely if $f \in \mathcal{C}^\infty\mathbb{R}^n)$ and $f = 0$ on H then

$$F^*\,(F(x), df(x)) = (x, d(f \circ F)(x)). \tag{7.15}$$

This certainly gives a pull-back map for functions on N^*H.

Consider the effect on ΩV of a linear transformation on V. If $L : V \longrightarrow W$ is an isomorphism of vector spaces then the dual map $L^* : W^* \longrightarrow V^*$ is also an isomorphism. These maps extend to the tensor powers, and hence to $\Lambda^k(V^*)$. Finally then this gives a linear isomorphism

$$L_* : \Omega V \longrightarrow \Omega W. \tag{7.16}$$

Here if $\nu : \Lambda^q(V^*) \longrightarrow \mathbb{R}$ is a V-density (an element of ΩV) then

$$L_*\nu(\gamma) = \nu(L^*\gamma),\ L^* : \Lambda^q W^* \longrightarrow \Lambda^q V^* \tag{7.17}$$

being the transpose isomorphism.

LEMMA 7.20. *Let $v_1, \dots, v_q$ and $w_1, \dots, w_q$ be bases for V and W, then $|v_1 \wedge \cdots \wedge v_q|$ and $|w_1 \wedge \cdots \wedge w_q|$ are bases for ΩV and ΩW and if*

$$Lv_i = \sum_{j=i}^{q} L_{ij} w_j \tag{7.18}$$

then

$$L_*|v_1 \wedge \cdots \wedge v_q| = |\det L_{ij}||w_1 \wedge \cdots \wedge w_q|. \tag{7.19}$$

PROOF. Let v_i^* and w_i^* be the dual bases of V^* and W^* respectively. Then the transpose if defined by

$$L^* w_i^* = \sum_{j=1}^{q} L_{ji} v_i^*. \tag{7.20}$$

This gives the identity

(7.21)
$$L_* v_1(\wedge \cdots \wedge v_q)(w_1^* \wedge \cdots \wedge w_q^*) = v_1 \wedge \cdots \wedge v_q(L^* w_1^* \wedge \cdots \wedge L^* w_q^*) = det(L_{ij})$$

which implies (7.19). □

EXERCISE 7.6. The transformation law (7.19) is the reason that densities appear in relation to integration. Let X be a manifold and for each $x \in X$ let $T_x X$ and $T_x^* X$ be respectively the tangent and cotangent spaces of x. Conventionally we set

$$\Omega_x X = \Omega(T_x^* X). \tag{7.22}$$

[This corresponds to the fact that, by definition, the form bundles $\Lambda^k X$ on X have fibres $\Lambda_x^k X = \Lambda^k(T_x^* X)$.] Thus if $x_1, \dots, x_n$ are local coordinates in X then $dx_1 \dots, dx_n$ is a basis for $T_x^* X$ locally and hence $|dx_1 \wedge \cdots \wedge dx_n|$ is a basis for $\Omega_x X$. As a coordinate change $x = X(y)$ induces the transformation (7.19) which is conventionally written

$$|dx_1 \wedge \cdots \wedge dx_n| = \left|\frac{\partial X}{\partial y}\right| |dy_1 \wedge \cdots \wedge dy_n|. \tag{7.23}$$

Thus if $\nu : X \longrightarrow \Omega X$ is a continuous section of compact support then $\int_X \nu$ is invariantly defined.

This explains the precise meaning of (7.3), since $|d\xi| \in (\Omega_{\text{fibre}})_x$ for each $x \in H$. Now F^* gives a fibre-preserving (linear bundle) isomorphism of N^*H which can be written

$$F^*(F'(0, x'), \xi dx_1) = (x', \xi \frac{\partial a}{\partial x_1}(0, x') dx_1). \tag{7.24}$$

Thus F induces an isomorphism

$$F^{\#} : S^M(N^*H) \otimes \Omega_{\text{fibre}} \longrightarrow S^M(N^*H) \otimes \Omega_{\text{fibre}} \tag{7.25}$$

by

$$F^{\#}(a(x',\xi)|d\xi|) = a\left((F'(0,x'),\xi/\frac{\partial a}{\partial x_1}(0,x')\right)\left|\frac{\partial a}{\partial x_1}(0,x')\right|^{-1}|d\xi|. \tag{7.26}$$

PROPOSITION 7.16. *Suppose $u \in I_c^m(\mathbb{R}^n, H)$, with $H = \{x_1 = 0\}$ and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism with $F(H) \subset H$ then, in terms of (7.3),*

$$\sigma_m(F^*u) = F^{\#}\sigma_m(u) \qquad \text{mod } S^{m+\frac{n}{4}-\frac{3}{2}}(N^*H)\otimes\Omega_{\text{fibre}}. \tag{7.27}$$

PROOF. We have already examined the transformation properties of conormal distributions and shown, in Proposition 3.11, that $F^*u \in I_c^m(\mathbb{R}^n, H)$. To prove this we decomposed F into a product

$$F = F_{(2)} \circ F_1 \circ F_0 \tag{7.28}$$

where F_0, F_1 and $F_{(2)}$ are of the respective forms

$$\begin{aligned} F_0(x_1,x') &= (x_1, f(x')) \quad f \text{ invertible} \\ F_1(x_1,x') &= (x_1\alpha'(x'),x') \quad \alpha'(x') \neq 0 \\ F_{(2)}(x_1,x') &= \left(x_1 + x_1^2 a'(x), x' + x_1 a''(x)\right). \end{aligned} \tag{7.29}$$

More precisely it was shown in (3.81) that

$$F_{(2)}^*u - u \in I_c^{m-1}(\mathbb{R}^n, H). \tag{7.30}$$

Notice that $F_{(2)}^{\#} = \text{Id}$ thus (7.30) reduces to (7.27) for $F = F_{(2)}$. Therefore it suffices to check (7.27) for $F = F_0$ and $F = F_1$ separately.

For F_0 given by (7.29)

$$F^{\#}(a(x',\xi)|d\xi|) = a(f(x'),\xi)|d\xi|. \tag{7.31}$$

If, corresponding to the definition of $\tilde{u}$ in (7.1),

$$u = 2^{-\frac{n}{2}}\pi^{-\frac{n}{4}-1}\int e^{ix_1\xi}a(\xi,x')d\xi \tag{7.32}$$

then

$$F_0^*u = 2^{-\frac{n}{2}}\pi^{-\frac{n}{4}-1}\int e^{ix_1\xi}a(\xi,f(x'))d\xi. \tag{7.33}$$

Thus $\sigma_m(F_0^*u) = a(\xi, f(x'))|d\xi| = F^{\#}(\sigma_m(u))$. This proves (7.27) for F_0.

For F_1 we have, from (7.32)

$$\begin{aligned} F_1^*u &= 2^{-\frac{n}{2}}\pi^{-\frac{n}{4}-1}\int e^{ix_1\alpha(x')\xi}a(\xi,x')d\xi \\ &= 2^{-\frac{n}{2}}\pi^{-\frac{n}{4}-1}\int e^{ix_1\Xi}a(\Xi/\alpha(x'),x')\frac{1}{|\alpha(x')|}d\Xi \end{aligned} \tag{7.34}$$

Thus again

$$\sigma_m(F_1^*u) = a(\Xi/\alpha(x'),x')\frac{1}{|\alpha(x')|}|d\xi_1| = F_1^{\#}(\sigma_m(u)). \tag{7.35}$$

This completes the proof of the proposition. □

Of course this shows that for *any* $\mathcal{C}^\infty$ hypersurface $H \subset \mathbb{R}^n$ we get a well-defined map

$$I_c^m(\mathbb{R}^n, H) \longrightarrow S^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \otimes \Omega_{\text{fibre}}/S^{m+\frac{n}{4}-\frac{3}{2}}, \tag{7.36}$$

by using (7.3) in any local coordinate in which $H = \{x_1 = 0\}$.

Lemma 7.21. *The linear map* (7.36) *gives rise to a short exact sequence*

$$\begin{aligned} 0 \longrightarrow I_c^{m-1}(\mathbb{R}^n, H) \longrightarrow I_c^m(\mathbb{R}^n, H) \\ \longrightarrow S_c^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \otimes \Omega_{\text{fibre}}/S_c^{m+\frac{n}{4}-\frac{3}{2}} \longrightarrow 0. \end{aligned} \tag{7.37}$$

where $S_c^M = \{a \in S^m; \pi[\text{supp}(a)] \subset\subset H,\ \pi : N^*H \longrightarrow H\}$.

Proof. That (7.36) is surjective is clear from (7.3). Similarly, again from (7.3), $\sigma_m(u) = 0$ means precisely that $\tilde{u}(\xi, x) \in S^{m+\frac{n}{4}-\frac{3}{2}}(\mathbb{R}^{n-1}, \mathbb{R})$, i.e. $u \in I_c^{m-1}(\mathbb{R}^n, H)$. □

Since $I^m(\mathbb{R}^n, H)$ is locally equal to $I_c^m(\mathbb{R}^n, H)$ we clearly have alternatively an exact sequence

$$\begin{aligned} 0 \longrightarrow I^{m-1}(\mathbb{R}^n, H) \hookrightarrow I^m(\mathbb{R}, H) \\ \longrightarrow S^{m+\frac{n}{4}-\frac{1}{2}}(N^*H) \otimes \Omega_{\text{fibre}}/S^{m+\frac{n}{4}-\frac{3}{2}} \longrightarrow 0 \end{aligned} \tag{7.38}$$

where $S^M(N^*H)$ is the space of symbols on the fibres of N^*H with support contained in $\pi^{-1}B$ for some closed subset $B \subset H$.

Having defined the symbol map (7.36) we next wish to consider the effect on the symbol of the two operations of pull-back and push-forward.

Recall that if $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^\infty$ map then the pull-back of conormal distributions associated to the hypersurface $H \subset \mathbb{R}^n$ is defined provided G is transversal to H. The latter condition is precisely

$$\begin{aligned} \forall\ y \in \mathbb{R}^n \text{ s.t. } G(y) \in H, dG^*h(y) \neq 0 \\ \text{if } h \in \mathcal{C}^\infty(\mathbb{R}^n), h = 0 \text{ on } H, dh \neq 0 \text{ at } G(y). \end{aligned} \tag{7.39}$$

As we have already noted, (7.39) implies that $G^{-1}(H) \subset \mathbb{R}^k$ is a $\mathcal{C}^\infty$ hypersurface for which G^*h is a defining function if $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a defining function for H. Furthermore

$$G_*(y, \eta dG^*h(y)) = (G(y), \eta dh(G(y)) \tag{7.40}$$

defines a $\mathcal{C}^\infty$ map

$$G_* : N^*G^{-1}(H) \longrightarrow N^*H \tag{7.41}$$

which is linear and invertible in each fibre. It therefore induces a map

$$\begin{aligned} G^\# : S^M(N^*H) \otimes \Omega_{\text{fibre}} \longrightarrow S^M(N^*G^{-1}(H)) \otimes \Omega_{\text{fibre}} \\ G^\#(a(x', \xi)|d\xi|) = 2^{-\frac{n-k}{2}}\pi^{-\frac{n-k}{4}-1}a(G(y'), \xi)|d(G_*)^*\xi|. \end{aligned} \tag{7.42}$$

If we introduce local coordinates (y_1, y') and (x_1, x') in the domain and range with $x_1 = h$, $y_1 = G^*h$, so

$$G(y_1, y') = (y_1, g(y')) \tag{7.43}$$

then the defining relation in (7.42) becomes simply

$$G^\#(a(x', \xi)|d\xi|) = a(G(y'), \xi)|d\eta| \tag{7.44}$$

in terms of the dual variables, ξ to x_1 and η to y_1.

PROPOSITION 7.17. *If $G : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^\infty$ map transversal to a hypersurface $H \subset \mathbb{R}^n$ then*

$$G^* : I^m(\mathbb{R}^n, H) \longrightarrow I^{m+\frac{n-k}{4}}(\mathbb{R}^k, G^{-1}(H)) \text{ and} \quad \sigma_{m+\frac{n-k}{4}}(G^*u) = G^\# \sigma_m(u). \tag{7.45}$$

PROOF. Using local coordinates (x_1, x'), (y_1, y') as in (7.42) we see that

$$u(x) = 2^{-\frac{n}{2}}\pi^{-\frac{n}{4}-1}\int e^{ix_1\xi}a(\xi, x')d\xi \Longleftrightarrow \quad G^*u(y) = 2^{-\frac{k}{2}}\pi^{-\frac{k}{4}-1}\int e^{iy_1\xi}a(\xi, g(y'))d\xi. \tag{7.46}$$

This immediately gives (7.45). □

A more serious concern is the effect of push-forward on conormal distributions. Thus suppose

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^k, \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^p,\ p = n - k \tag{7.47}$$

is projection onto the first k factors

$$F(x_1, \dots, x_n) = (x_1, \dots, x_k). \tag{7.48}$$

We shall, as usual, divide the coordinates into $x = (y, \theta)$, $y \in \mathbb{R}^k$ and $\theta \in \mathbb{R}^p$. If $H \subset \mathbb{R}^n$ is a hypersurface we need a condition to guarantee that $F_*(u)$ is conormal with respect to some hypersurface if $u \in I_c^m(\mathbb{R}^n, H)$.

The *local* condition we impose is

$$H \text{ is at most simply tangent to the fibres of } F. \tag{7.49}$$

If we let

$$C_H = \{x \in H;\ \text{the fibre of } F \text{ through } x \text{ is tangent to } H\} \tag{7.50}$$

be the critical set of H with respect to the map F then we can express (7.49) in terms of any defining function, $h \in \mathcal{C}^\infty(\mathbb{R}^n)$, of H. Thus (7.50) can be written

$$C_H = \{x \in H; d_\theta h(y, \theta) = 0, x = (y, \theta)\}. \tag{7.51}$$

The condition of simple tangency is just

$$d_\theta\left(\frac{\partial h}{\partial \theta_j}\right)\ j = 1, \dots, p \quad \text{are linearly independent at each point of } C_H. \tag{7.52}$$

LEMMA 7.22. *Under the non-degeneracy condition* (7.49)*, or equivalently* (7.52)*, $C_H \subset \mathbb{R}^n$ is a $\mathcal{C}^\infty$ submanifold of codimension $p + 1$ and*

$$F_C : C_H \longrightarrow \mathbb{R}^k,\ F_C = F|_{C_H}, \tag{7.53}$$

is locally a diffeomorphism onto a hypersurface.

PROOF. This is just the implicit function theorem. Thus at $(\bar{y}, \bar{\theta}) \in C_H$ the independence condition (7.52) means that the p equations

$$\frac{\partial h}{\partial \theta_j}(y, \theta) = 0 \qquad j = 1, \dots, p \tag{7.54}$$

have a unique local solution near $(\bar{y}, \bar{\theta})$ as equations for θ, and this is given by a $\mathcal{C}^\infty$ map

$$\theta = \Theta(y) \Longleftrightarrow \frac{\partial h}{\partial \theta}(y, \theta) = 0. \tag{7.55}$$

Thus $C_H = \{(y, \Theta(y))\} \cap H$. Now if

$$\tilde{h}(y) = h(y, \Theta(y)) \tag{7.56}$$

then $d\tilde{h}(\bar{y}) \neq 0$ since $dh \neq 0$ and $d_\theta h = 0$. Thus the range of (7.53) is locally

$$F(C_H) = \{y \in \mathbb{R}^n; y \text{ near } \bar{y}, \tilde{h}(y) = 0\}. \tag{7.57}$$

This proves the lemma. □

Suppose we to add to (7.52) the condition that (7.53) be globally a diffeomorphism onto a $\mathcal{C}^\infty$ hypersurface $F(C_H)$ (or just work locally). Then there is again a map

$$F_* : N^*_{C_H} H \longrightarrow N^*[F(C_H)] \tag{7.58}$$

given by

$$(x, \xi dh) \longmapsto (F(x), \xi d\tilde{h}(y)),\ x = (y, \theta). \tag{7.59}$$

To state the result for push-forward we need to take note of two extra objects.

First note that to define the push-forward, $F_*(u)$, of $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$, we need to use the density $|d\theta|$ on the fibres of F. Using this density we can construct a function on $N^*[F(C_H)]$ as follows. At any point $\bar{x} = (\bar{y}, \bar{\theta})$ of C_H let $v_1, \dots, v_p$ be any basis of $\mathbb{R}^p$, the fibre of F through $\bar{x}$, such that

$$|d\theta|(v_1 \wedge \cdots \wedge v_p) = 1. \tag{7.60}$$

For example $\partial_{\theta_1}, \dots, \partial_{\theta_p}$. Then set

$$\mu(\bar{y}, \eta d\tilde{h}) = |\eta|^{-\frac{p}{2}} \left| \frac{1}{2} \det v_i v_j h(\bar{y}, \bar{\theta}) \right|^{\frac{1}{2}}. \tag{7.61}$$

Notice that the matrix $v_i v_j h$ is well-defined at $(\bar{y}, \bar{\theta})$ for any defining function h since if V_i are $\mathcal{C}^\infty$ vector fields extending the v_i, i.e. with $V_i(\bar{y}, \bar{\theta}) = v_i$, then $V_i V_j h(\bar{y}, \bar{\theta})$ is well-defined, independent of the extension. Moreover if $h' = \alpha(y, \theta) h$ is another defining function then

$$V_i V_j \alpha h(\bar{y}, \bar{\theta}) = \alpha V_i V_j h(\bar{y}, \bar{\theta}). \tag{7.62}$$

In particular the right hand side of (7.61) is well-defined once the multiple $\eta\tilde{h}$ is prescribed.

The second object is similar, being another function on $N^*[F(C_H)]$. Namely, consider the signature of the Hessian of the symmetric matrix $H_v = v_i v_j h(\bar{y}, \bar{\theta})$. Our convection for the signature is the number of positive minus the number of negative eigenvalues. Once the sign of h is fixed, notice that the signature is actually independent of the choice of the v_i's since if

$$v'_i = \sum_{j=1}^{p} Q_{ij} v_j \tag{7.63}$$

is another choice of basis then $H_{v'} = Q \cdot H_v \cdot Q^t$. Thus

$$\operatorname{sgn}(H, F)(\bar{y}, \eta d\tilde{h}) = \operatorname{sgn} v_i v_j h \tag{7.64}$$

is actually a locally constant function with integer values on $N^*[F(C_H)]\setminus 0$. It also satisfies

$$\operatorname{sgn}(H,F)(y',-\eta) = -\operatorname{sgn}(H,F)(y',\eta)\ \forall\ (y',\eta)\in N^*[F(C_H)]\setminus 0. \tag{7.65}$$

We can now state the result for push-forward:

Proposition 7.18. *If $H\subset\mathbb{R}^n$ satisfies* (7.49) *for the map* (7.48) *and F_C in* (7.53) *is a diffeomorphism onto a hypersurface then the push-forward operation gives a map*

$$F_*: I_c^m(\mathbb{R}^n,H)\longrightarrow I^{m-\frac{n-k}{4}}\left(\mathbb{R}^k,F(C_H)\right) \tag{7.66}$$

with the symbolic property

$$\sigma(F_*u) = F_\#\sigma(u) \tag{7.67}$$

where

$$F_\#(a|d\xi|)(y,\xi) = 2^{-\frac{n-k}{2}}\pi^{\frac{n-k}{4}-1}e^{-i\frac{1}{4}\pi\operatorname{sgn}(H,F)}a(F_c^{-1}(y),\eta)\frac{1}{\mu(y,\eta)}|d\eta| \tag{7.68}$$

in local coordinates in which $\tilde h = y_1$.

The proof of this result is essentially equivalent to the 'principal of stationary phase.' We shall not describe this fully here but refer instead to the elegant treatment by Hörmander in [**2**].

The Morse Lemma allows a hypersurface satisfying the non-degeneracy condition (7.52) to be brought into a simple normal form locally, in a way that preserves the fibration. For completeness sake we give a proof, as usual by the homotopy method.

Lemma 7.23. *If H is a hypersurface in $\mathbb{R}^n$ satisfying the non-degeneracy condition* (7.52) *for the map F in* (7.48) *at $(\bar y,\bar\theta)$, then there is a fibre-preserving transformation*

$$\begin{gathered} G(y,\theta) = (y,\Theta(y,\theta))\ \text{such that}\ \Theta(\bar y,\bar\theta)=0\ \text{and}\\ G(H) = \{y_1+\sum_{i=1}^{q}\theta_i^2-\sum_{i=q+1}^{p}\theta_i^2=0\}. \end{gathered} \tag{7.69}$$

Proof. From the proof of Lemma 7.22 we can make a preliminary fibre-preserving transformation so that $G(\bar y,\bar\theta)=(\bar y,0)$ and $C_{G(H)}=\{y_1=0,\theta=0\}$ locally. Thus we can suppose that

$$h(y,\theta) = \tilde h(y)+g(y,\theta) \tag{7.70}$$

where $g(y,0)=0$ and $d_\theta g(y,0)=0$. By the non-degeneracy assumption the Hessian of g, $\partial^2 g/\partial\theta^2$ must be invertible.

By a linear change of θ variable we can certainly arrange that

$$\partial^2 g/\partial\theta_i\pi_j(\bar y,0) = 2\epsilon_i\delta_{ij},\ i,j=1,\dots,p,\ \epsilon_i=1, i\le q, \epsilon_i=-1, i>q. \tag{7.71}$$

Here of course, $2q-p$ is the signature of the Hessian. Once (7.71) holds the one-parameter family of functions

$$h_t = \tilde h + tg(y,\theta)+(1-t)[\sum_{i\le q}\theta_i^2-\sum_{i>q}\theta_i^2],\ 0\le t\le 1 \tag{7.72}$$

satisfies (7.71) for all $t \in [0,1]$. Thus the derivatives $\partial_{\theta_i} h_t$, $i = 1, \dots, p$ are independent at $\theta = 0$ for y near $\bar{y}$. Since the t-derivative vanishes to second order at $\theta = 0$ we can find p $\mathcal{C}^\infty$ functions of (t, y, θ), such that

$$\frac{dh_t}{dt} = -\sum_{i=1}^{p} V_i(t, y, \theta)\partial_{\theta_i} h_t(y, \theta). \tag{7.73}$$

If we regard $(V_1, V_1, \dots, V_p)$ as a vector field on the fibres of F,

$$V = \sum_{i=1}^{p} V_i(t, y, \theta)\partial_{\theta_i} \tag{7.74}$$

and solve the differential equations

$$\frac{d\Theta_i(t, y, \theta)}{dt} = V_i(t, y, \Theta),\ \Theta(0, y, \theta) = \theta,\ i = 1, \dots, p \tag{7.75}$$

we define a 1-parameter family of local fibre-preserving diffeomorphisms, G_t such that

$$\frac{d}{dt} G_t^* h_t = 0,\ G_0 = \mathrm{Id} \Longrightarrow G_1^* h = h_0. \tag{7.76}$$

Since h_0 is of the form (7.69) this proves the lemma. □

Proof. Proof of Proposition 7.18 The problem is certainly a local one, so we can assume (recalling Proposition 5.14) that $u \in I^m(\mathbb{R}^n, H)$ has its support in some conveniently small neighbourhood of $(\bar{y}, \bar{\theta}) \in C_H$. In particular we can suppose its support to be within the domain of the fibre-preserving transformation found in Lemma 7.23. Thus

$$u(y, \theta) = 2^{-\frac{n}{2}} \pi^{-\frac{n}{4}-1} \int e^{i(y_1 + |\theta'|^2 - |\theta''|^2)\tau} a(y, \theta', \theta'') d\tau \tag{7.77}$$

where $\theta = (\theta', \theta'')$. This change of variables does change the fibre-measure involved in the push-forward. Namely

$$F_*(u) = \int_{\mathbb{R}^p} u(y, \theta) J(y, \theta) d\theta,\ J(y, \theta) = |\partial_\theta \Theta(y, \theta)|. \tag{7.78}$$

In any case this means that

$$F_*(u)(y) = 2^{-\frac{n}{2}} \pi^{-\frac{n}{4}-1} \int_{\mathbb{R}} e^{iy_1\tau} b(y, \tau) d\tau \tag{7.79}$$

if

$$b(y, \tau) = \int_{\mathbb{R}^p} e^{i(|\theta|^2 - |\theta''|^2)\tau} a(y, \theta, \tau) J(y, \theta) d\theta. \tag{7.80}$$

To work out the form of b it suffices to integrate out the θ variables successively.

Lemma 7.24. *If $a \in S^M(\mathbb{R}^\ell \times \mathbb{R}, \mathbb{R})$ has support in $\{|(y, r)| \le 1\} \times \mathbb{R}$ then*

$$a_\pm(y, \tau) = \int_{\mathbb{R}} e^{\pm i\tau r^2} b(y, r, \tau) dr \in S^{M-\frac{1}{2}}(\mathbb{R}^\ell; \mathbb{R}) \tag{7.81}$$

and if $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ *has* $\phi(\tau) = 1$ *in* $|\tau| \le 1$

$$a_\pm(y,\tau) - (1-\phi(\tau))|\tau|^{-\frac12} e^{\mp \operatorname{sgn}(\tau) i \frac14 \pi} \sum_{k<j} \frac{(\pm i)^k}{k!\tau^k} \partial_r^{2k} b(y,0,\tau) \tag{7.82}$$

$$\in S^{M-\frac12-j}(\mathbb{R}^\ell, \mathbb{R}) \ \forall\ j. \tag{7.83}$$

Proof. Certainly $a_\pm$ is $\mathcal{C}^\infty$, so we only need to consider $(1-\phi(\tau))a_\pm$ and we can therefore assume that $|\tau| \ge 1$ on the support of b. Directly by estimating (7.81) we have

$$|a_\pm(y,\tau)| \le C|\tau|^M \tag{7.84}$$

where this just depends on the order of b. Choose $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ to be identically equal to 1 in $[-1,1]$. Then, from the assumption on the support of b, (7.81) continues to hold if $\rho(r)$ is inserted into the integrand. Next replace b by its Taylor series in r to order $2j$. The error term gives a contribution to $a_\pm$ of the form

$$a_{\pm,j}(y,\tau) = \int_{\mathbb{R}} e^{\pm i\tau r^2} r^{2j} b_j(y,r,\tau)\rho(r)dr \tag{7.85}$$

where b_j is still a symbol of order M. Using the identity

$$re^{\pm i\tau r^2} = (\pm i\tau)^{-1}\frac12 \partial_r e^{\pm i\tau r^2} \tag{7.86}$$

and integration by parts j times allows $a_{\pm,j}$ to be written in the form (7.81) with b replaced by a symbol of order $M-j$. Moreover differentiation by y leaves the form unchanged and differentiation by τ either lowers the symbol order by 1 or adds a factor of r^2 which, by integration by parts, also lowers the symbol order by 1. It therefore follows that

$$b_{\pm,j} \in S^{M-j}(\mathbb{R}^\ell;\mathbb{R}). \tag{7.87}$$

This does not prove (7.81), but allows us to consider only the finite terms obtained from the Taylor series, we can suppose therefore that $b = r^k b^{(k)}(y,\tau)/k!$. If k is odd the resulting contribution to (7.81) is clearly zero so consider an individual term

$$r^{2k} b^{(2k)} b(y,\tau)/(2k)!. \tag{7.88}$$

Integration by parts shows that this contributes to $a_\pm$ a term

$$a_{\pm,k} = \int_{\mathbb{R}} e^{\pm i\tau r^2} r^{2j} \frac{(\pm i)^k}{k!\tau^k} b^{(2k)}(y,r,\tau)\rho(r)dr \tag{7.89}$$

since any differentiation of ρ makes a contribution which is rapidly decreasing as $\tau \to \infty$. Thus to prove (7.83) so we only need consider the particular case where b is independent of r :

$$a_{\pm,0}(y,\tau) = \int_{\mathbb{R}} e^{\pm i\tau r^2} b(y,0,\tau)\rho(r)dr. \tag{7.90}$$

Since the support of the integrand is bounded we can obtain this as a limit where the exponential is made to be strongly decreasing at infinity

$$a_{\pm,0}(y,\tau) = \lim_{\epsilon\to 0, \pm\epsilon\tau>0} \int_{\mathbb{R}} e^{\pm(1+i\epsilon)i\tau r^2} b(y,0,\tau)\rho(r)dr. \tag{7.91}$$

Taking $\tau > 0$ for simplicity the exponential, for $\pm\epsilon > 0$, is rapidly decreasing as $|r| \to \infty$. Thus the two terms

$$a_{\pm,0}(y,\tau) = \lim_{\pm\epsilon\downarrow 0}[\alpha_\pm(\epsilon,\tau) + \beta_\pm(\epsilon,\tau)]b(y,0,\tau)\ \tau > 0 \tag{7.92}$$

make sense where

$$\beta_\pm(\epsilon,\tau) = \int_{\mathbb{R}} e^{i(1+i\epsilon)\tau r^2}(1-\rho(r))dr. \tag{7.93}$$

Again integrating by parts in r gives

$$\beta = \int_{\mathbb{R}} e^{i(1+i\epsilon)\tau r^2} W^k(1-\rho(r))dr\ \forall\ k,\ W = -\partial_r[(1+i\epsilon)\tau r]^{-1}. \tag{7.94}$$

This is uniformly, in $\pm\epsilon \in [0,1]$, rapidly decreasing as $\tau \to \infty$ and so contributes a trivial term to $a_{\pm,0}$.

By changing variable of integration to $\tau^{\frac{1}{2}}r$ note that

$$\lim_{\pm\epsilon\downarrow 0}\alpha_\pm(\epsilon,\tau) = |\tau|^{-\frac{1}{2}}\lim_{\pm\epsilon\downarrow 0}\alpha_\pm(\epsilon,1). \tag{7.95}$$

Thus we only need to carry out a simple computation to find that

$$\alpha_\pm = \lim_{\pm\epsilon\to 0}\int_{\mathbb{R}} e^{\pm i(1+i\epsilon)r^2}dr = \pi^{\frac{1}{2}}e^{\mp i\frac{1}{4}\pi}. \tag{7.96}$$

The discussion above shows that the limit exists and its real part

$$\lim_{\eta\downarrow 0}\int_{\mathbb{R}}\cos(r^2)e^{-\eta r^2}dr \geq 0. \tag{7.97}$$

Squaring the integral in (7.96) and introducing polar coordinates gives

$$\alpha_\pm^2 = \lim_{\pm\epsilon\downarrow 0}\int_{[0,2\pi]}\int_0^\infty e^{\pm i(1+i\epsilon)R^2}RdRd\phi = \lim_{\pm\epsilon\downarrow 0}\pm\pi[i(1+i\epsilon)]^{-1} = \mp\pi i. \tag{7.98}$$

In view of (7.97) this proves (7.96). This in turn proves (7.83) with the symbol space of order $M-j$ on the right. However since this is true for all j and the next term is of order $M-j-\frac{1}{2}$ the result as stated follows, completing the proof of Lemma 7.24. □

Applying Lemma 7.24 repeatedly to (7.77) gives (7.68) in case $\mu = 1$. The general case follows once the change of variables leading to (7.78) is taken into account. Indeed at C_H we find from

$$\mu(y,dh)^{-1} = |\frac{\partial\Theta(y,\theta)}{\partial\theta}| = J(y,\theta) \tag{7.99}$$

so this factor just corresponds to the Jacobian in Lemma 7.24. □

CHAPTER 8

The wave group and the scattering kernel

The forward fundamental solution for the perturbed wave equation, the existence of which is shown in Chapter 6, is used to solve the Cauchy problem. Then, using the modified Radon transform of Lax and Phillips, we define the scattering kernel and discuss some of its basic properties.

Consider again the Cauchy problem:

$$P_V u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^n \\ u|_{t=0} = u_0 \text{ and } D_t u|_{t=0} = u_1. \tag{8.1}$$

THEOREM 8.3. *If u_0, $u_1 \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ there is a unique $u \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfying (8.1) and the solution satisfies*

$$\operatorname{supp}(u) \cap [-T, T] \times \mathbb{R}^n \subset\subset \mathbb{R} \times \mathbb{R}^k \quad \forall\ T \in \mathbb{R}. \tag{8.2}$$

PROOF. We can easily find $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n)$ which satisfies the initial conditions and the equation in Taylor series at $t = 0$. This only involves choosing the Taylor series of $\tilde{u}$ to be

$$\tilde{u} \sim u_0 + (it)u_1 + \sum_{j\geq 2} \frac{(it)^j}{j!} u_j \\ \text{with } u_{j+2} = (\Delta + V)u_j \ \forall\ j \geq 0. \tag{8.3}$$

Then the difference $v = u - \tilde{u}$ should satisfy

$$P_V v = f = -P_V \tilde{u},\ v|_{t=0} = 0 \text{ and } D_t v|_{t=0} = 0. \tag{8.4}$$

Here $f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$ has all derivatives zero at $t = 0$. It can therefore be divided into a 'forward' and a 'backward' part:

$$f = f_+ + f_-,\ f_\pm \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n),\ f_\pm(t, x) = 0 \text{ in } \pm t < 0. \tag{8.5}$$

We then construct $v = v_+ + v_-$ by using Proposition 6.15 to solve the two forcing problems:

$$P v_\pm = f_\pm \text{ with } \operatorname{supp}(v_\pm) \subset \{\pm t \geq 0\}. \tag{8.6}$$

This can be done using the forward and backward fundamental solutions:

$$v_\pm(t, x) = \int\int E_+(\pm(t - s), x, x') f_\pm(s, x') ds dx'. \tag{8.7}$$

Notice that, from (8.5) and the support properties of E_+, the distributional pairing here is well defined and the $v_\pm \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (8.6). Thus

$$u = \tilde{u} + v_+ + v_- \text{ solves (8.1)}. \tag{8.8}$$

To get the uniqueness consider the difference, w, of two smooth solutions of (8.1). Then $w \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$ must satisfy

$$P_V w = 0, \; w|_{t=0} = 0, \; D_t w|_{t=0} = 0. \tag{8.9}$$

It follows that the Taylor series of w at $t = 0$ must vanish. Thus w can be divided into forward and backward parts:

$$w = w_+ + w_- \text{ with } \operatorname{supp}(w_\pm) \subset \{\pm t \geq 0\}. \tag{8.10}$$

Consider the forward part of w. It must satisfy

$$P_V w_+ = 0 \text{ and } \operatorname{supp}(w_+) \subset \{t \geq 0\} \tag{8.11}$$

The uniqueness part of Proposition 6.15 therefore shows that $w_+ = 0$. The same argument applies to w_-, so $w = 0$ and uniqueness holds. □

The unique solvability of the Cauchy problem (8.1) allows us to define the wave group, which is the 1-parameter family of operators

$$\begin{gathered} U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t,\cdot) \\ D_t u(t,\cdot) \end{pmatrix} \quad \forall\, t \in \mathbb{R} \\ U(t) : \mathcal{C}_c^\infty(\mathbb{R}^n) \times \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^n) \times \mathcal{C}_c^\infty(\mathbb{R}^n). \end{gathered} \tag{8.12}$$

As the name indicates this is a *group* of operators. When we wish to emphasize the dependence on the potential we shall use the notation $U_V(t)$.

Lemma 8.25. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$*

$$U_V(0) = \mathrm{Id} \; \textit{and} \; U_V(t) \cdot U_V(s) = U_V(t+s) \; \forall\, t, s \in \mathbb{R}. \tag{8.13}$$

Proof. The group property follows from the t-translation invariance of P_V and the uniqueness of the solution of (8.1). Thus if $u \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies $P_V u = 0$ then

$$u_s(t,x) = u(t+s,x) \text{ satisfies } P_V(u_s) = 0. \tag{8.14}$$

If u is the solution of (8.1) then for any $s \in \mathbb{R}$

$$\begin{gathered} P_V(u_s) = 0, \\ u_s|_{t=0} = u(s,\cdot) \text{ and } D_t u_s|_{t=0} = D_t u(s,\cdot). \end{gathered} \tag{8.15}$$

This just means that u_s is the solution of (8.1) with initial data $U(s)\binom{u_0}{u_1}$, so

$$U(t)\left(U(s)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right) = \begin{pmatrix} u_s(t,\cdot) \\ D_t u_s(t,\cdot) \end{pmatrix} = U(t+s)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \tag{8.16}$$

□

For the moment we shall revert to the 'trivial case' of $V \equiv 0$. Since the support of u satisfying (8.1) for $u_0, u_1 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is compact in x for all finite t we can take its Radon transform and set

$$v(t,s,\omega) = \int\limits_{x\cdot\omega = s} u(t,x) dH_x = Ru(t,\cdot)(s,\omega). \tag{8.17}$$

Recall the important intertwining property (2.17) of the Radon transform. Differentiating (8.17) and using this we find

$$D_t^2 v = \int\limits_{x\cdot\omega = s} D_t^2 u(t,x) dH_x = R(\Delta u)(t,\cdot) = D_s^2 v. \tag{8.18}$$

Thus v satisfies the wave equation in one space variable

$$(D_t^2 - D_s^2)v(t,s,\omega) = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_s \times \mathbb{S}^{n-1}_\omega \tag{8.19}$$

in which ω is purely a parameter. A standard result in elementary distribution theory characterizes all solutions of (8.19) in the form

$$v(t,s,\omega) = f(s-t,\omega) + g(s+t,\omega). \tag{8.20}$$

Here $f, g \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ are determined up to a constant. Indeed from (8.20)

$$\begin{aligned} v(0,s,\omega) &= f(s,\omega) + g(s,\omega) \\ D_t v(0,s,\omega) &= -D_s f(s,\omega) + D_s g(s,\omega). \end{aligned} \tag{8.21}$$

Differentiating the first equation allows us to find the derivatives of f and g

$$\begin{aligned} D_s f(s,\omega) &= \frac{1}{2}[D_s v(0,s,\omega) - D_t v(0,s,\omega)] \\ D_s g(s,\omega) &= \frac{1}{2}[D_s v(0,s,\omega) + D_t v(0,s,\omega)]. \end{aligned} \tag{8.22}$$

Now by the definition, (8.17), of v we have

$$v(0,s,\omega) = (Ru_0)(s,\omega), \;\; D_t v(0,s,\omega) = (Ru_1)(s,\omega). \tag{8.23}$$

Thus (8.22) becomes

$$D_s f(s,\omega) = \frac{1}{2}\left[D_s(Ru_0)(s,\omega) - Ru_1(s,\omega)\right]. \tag{8.24}$$

We have dropped the reference to g here because of the symmetry properties of the Radon transform. Thus, from (8.17)

$$v(t,-s,-\omega) = v(t,s,\omega). \tag{8.25}$$

Inserting this into (8.20) shows that

$$f(s-t,\omega) + g(s+t,\omega) \equiv f(-s-t,-\omega) + g(-s+t,-\omega). \tag{8.26}$$

Differentiating with respect to s we conclude that

$$D_s g(s,\omega) = D_s\left[f(-s,-\omega)\right]. \tag{8.27}$$

[We cannot directly conclude that $g(s,\omega) = f(-s,-\omega)$ only because of the indeterminacy of f and g up to a relative constant.]

From now on we shall confine our attention to the case that $n \geq 3$ is *odd* unless otherwise explicitly noted. The reason for doing so is that the Radon inversion formula then takes the simple form:

$$\phi(x) = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} (D_s^{n-1} R\phi)(x\cdot\omega,\omega)d\omega. \tag{8.28}$$

In view of this and (8.24) we *define* the modified Radon transform of Lax and Phillips to be

$$\begin{aligned} LP\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} &= 2^{\frac{1}{2}}(\pi)^{\frac{n-1}{2}} D^{\frac{n+1}{2}} f(s,\omega) \\ &= 2^{\frac{1}{2}}(\pi)^{\frac{n-1}{2}}\left\{D^{\frac{n+1}{2}}(Ru_0)(s,\omega) - D_s^{\frac{n-1}{2}}(Ru_1)(s,\omega)\right\}. \end{aligned} \tag{8.29}$$

LEMMA 8.26. *The modified Radon transform of Lax and Phillips is an injective map*

$$\text{LP} : \mathcal{C}_c^\infty(\mathbb{R}^n) \times \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1}) \tag{8.30}$$

(for $n \geq 3$ odd) which intertwines the free wave group and the translation group:

$$\text{LP} \cdot U_0(t) = T_t \cdot LP,\ T_t v(s,\omega) = v(s-t,\omega). \tag{8.31}$$

PROOF. From the definition, (8.17), of v we certainly see that the transform, v, corresponding to $U_0(t')\binom{u_0}{u_1}$ is just

$$v(t+t',s,\omega) = \int\limits_{x\cdot\omega=s} u(t+t',x)dH_x. \tag{8.32}$$

The decomposition (8.20) is

$$v(t+t',s,\omega) = f(s-t-t',\omega) + g(s+t+t',\omega) \tag{8.33}$$

so that (8.29) gives

$$\text{LP}\left[U(t')\begin{pmatrix}u_0\\u_1\end{pmatrix}\right] = 2^{\frac{1}{2}}(\pi)^{\frac{n-1}{2}} D_s^{\frac{n+1}{2}} f(s-t',\omega) = T_{t'} LP\begin{pmatrix}u_0\\u_1\end{pmatrix}. \tag{8.34}$$

Thus we have proved (8.31). The injectivity follows from the inversion formula (8.28) and (8.27). More precisely, $D^{\frac{n+1}{2}} f = 0 \Longrightarrow D_s^{\frac{n+1}{2}} g = 0$ (since $\frac{n-1}{2} > 1$) so from (8.19), for $u_0, u_1 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\text{LP}\begin{pmatrix}u_0\\u_1\end{pmatrix} = 0 \Longrightarrow D_s^{\frac{n+1}{2}} v = 0 \Longrightarrow u_0 = u_1 = 0. \tag{8.35}$$

□

Just as for the Radon transform itself, LP is *not* surjective as a map (8.30). Nevertheless we show, in Proposition 8.21 below, that it extends to an isomorphism of appropriate Hilbert (Sobolev) spaces.

Another way of looking at (8.31) is to observe that if u is the solution of (8.1) and

$$k(t,s,\omega) = LP \cdot U_0(t)\begin{pmatrix}u_0\\u_1\end{pmatrix} \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}) \tag{8.36}$$

then $k(t,s,\omega) = k_0(s-t,\omega)$ is a solution of the first order differential equation

$$(D_t + D_s)k(t,s,\omega) = 0 \text{ in } \mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}. \tag{8.37}$$

Of course this discussion has all been concerned with the free wave equation, i.e. the trivial case $V \equiv 0$. Let us now see what happens if we allow a general potential $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. We can still define k by analogy with (8.36):

$$k_V(s,t,\omega) = LP \cdot U_V(t)\begin{pmatrix}u_0\\u_1\end{pmatrix} \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}), \tag{8.38}$$

only now (8.37) will not hold. From (8.29) we have

$$\begin{aligned}&(D_t + D_s)k_V(t,s,\omega)\\ &= 2^{\frac{1}{2}}(\pi)^{\frac{n-1}{2}}\left\{-D_s^{\frac{n-1}{2}} R D_t^2 u(t,\cdot) + D_s^{\frac{n+3}{2}} R u(t,\cdot)\right\}\\ &= -2^{\frac{1}{2}}(\pi)^{\frac{n-1}{2}} D_s^{\frac{n-1}{2}} R[V(\cdot)u(t,\cdot)].\end{aligned} \tag{8.39}$$

Now with $k_0(s,\omega) = LP\binom{u_0}{u_1}$, as before, (8.21) and (8.23) show that

$$D_s^{n-1} R u_0 = D_s^{n-1} [f(s,\omega) + g(s,\omega)] = D_s^{n-1} [f(s,\omega) + f(-s,-\omega)]. \tag{8.40}$$

Then from (8.28)

$$\begin{aligned} u_0(x) &= \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} D_s^{n-1}(f(s,\omega) + f(-s,-\omega))|_{s=x\cdot\omega}\, d\omega \\ &= \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} (D_s^{n-1} f)(x\cdot\omega,\omega) d\omega \\ &= 2^{\frac{1}{2}} (\pi)^{\frac{n-1}{2}} \int\limits_{\mathbb{S}^{n-1}} (D_s^{\frac{n-3}{2}} k_0)(x\cdot\omega,\omega) d\omega \end{aligned} \tag{8.41}$$

since n is odd. A similar discussion allows us to recover $u_1(x) = D_t u(0,x)$ as

$$u_1(x) = 2^{\frac{1}{2}} (\pi)^{\frac{n-1}{2}} \int\limits_{\mathbb{S}^{n-1}} (D_s^{\frac{n-1}{2}} k_0(x\cdot\omega,\omega)) d\omega. \tag{8.42}$$

Applying (8.41) to (8.38) for each value of t, using (8.29), gives

$$u(t,x) = 2^{\frac{1}{2}} (\pi)^{\frac{n-1}{2}} \int\limits_{\mathbb{S}^{n-1}} (D_s^{\frac{n-3}{2}} k_V)(t, x\cdot\omega,\omega) d\omega, \tag{8.43}$$

where u is the solution of (8.1).

Inserting this inversion formula into (8.39) gives

$$(D_t + D_s) k_V(t,s,\omega) + V_{\mathrm{LP}} \cdot k_V(t,s,\omega) = 0 \tag{8.44}$$

where V_{LP} is an operator on $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$:

$$V_{\mathrm{LP}} = \frac{1}{2(2\pi)^{n-1}} D_s^{\frac{n-1}{2}} \cdot R \cdot V \cdot R^t D_s^{\frac{n-3}{2}}, \tag{8.45}$$

the operator V being multiplication by V.

Lemma 8.27. *If* $\operatorname{supp}(V) \subset \{|x| \le \rho\}$ *the operator* V_{LP} *defined by* (8.45) *has Schwartz kernel* $V_{\mathrm{LP}}(s,\omega,s',\omega')$ *supported in the region*

$$\operatorname{supp}(V_{\mathrm{LP}} \subset \left\{(s,\omega',s',\omega) \in \mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{R}\times\mathbb{S}^{n-1}; |s|,|s'| \le \rho\right\}. \tag{8.46}$$

Proof. For any $k \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ the $\mathcal{C}^\infty$ function $v = V\cdot R^t \cdot D_s^{\frac{n-3}{2}} k$ has support in $|x| \le \rho$ since V has support there. Applying Proposition 2.4 we conclude that $V_{\mathrm{LP}} k$ always has support in $|s| \le \rho$. The same argument applies to the transpose of V_{LP}, so (8.46) follows. □

The equation (8.44) is very similar to the original equation $P_V u = 0$. For example

$$k_V(t,s,\omega) \text{ satisfies } (D_t + D_s) k_V = 0 \text{ in } |s| > \rho \tag{8.47}$$

where $\operatorname{supp}(V) \subset \{|x| \le \rho\}$. Thus k_V satisfies the free equation to the left and right of the 'potential' V_{LP}; see Figure 1.

Consider the Cauchy problem for the transformed equation (8.44). From (8.46) it follows that

$$V_{\mathrm{LP}} : \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}). \tag{8.48}$$

from -200 0 to 200 0 ¡4pt¿ [.2,.67] from 0 -180 to 0 180 -130 -150 150 130 / -40 60 40 -20 / axes ratio 1:3 280 degrees from 241 12 center

FIGURE 1. Support and singular support of $s' = \frac{1}{2}\rho$

Then we look for a distribution satisfying

$$(D_t + D_s + V_{\mathrm{LP}})E_{\mathrm{LP}}(t, s, \omega; s', \theta) = 0 \\ E_{\mathrm{LP}}(0, s, \omega; s', \theta) = \delta(s - s')\delta_\theta(\omega). \tag{8.49}$$

Naturally we expect this to correspond to a solution of the perturbed wave equation. Applying (8.41) and (8.42) the initial data should be

$$u_0(x) = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}} D_{s'}^{\frac{n-3}{2}} \delta(x \cdot \theta - s') \text{ and} \\ u_1(x) = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}} D_{s'}^{\frac{n-1}{2}} \delta(x \cdot \theta - s'). \tag{8.50}$$

Certainly we can find a solution to (8.1) with the initial data (8.50). From the arguments used to solve (1.33) we know that the solution is of the form

$$u = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}} D_t^{\frac{n-3}{2}} \delta(x \cdot \theta - s' + t) + w_+ + w_-, \\ w_\pm \in I^{\frac{3n}{4}-3}(\mathbb{R}^{n+1}, \{t = \pm(s' - x \cdot \theta)\}) \tag{8.51}$$

and that the $w_\pm$ have compact x-supports for all bounded time. To get from this to the solution of (8.49) we need to apply the operator LP, so we pause to consider its mapping properties.

PROPOSITION 8.19. *The modified Radon transform* (8.29) *extends by continuity to a linear operator*

$$\mathrm{LP} : [S^{-\frac{n}{2}-\frac{3}{2}+}(\mathbb{R}^n)]' \times [S^{-\frac{n}{2}-\frac{1}{2}+}(\mathbb{R}^n)]' \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R} \times \mathbb{S}^{n-1}). \tag{8.52}$$

The operator

$$\mathrm{LP}^{-1} = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}} \left(R^t \cdot D_s^{\frac{n-3}{2}}, R^t \cdot D_s^{\frac{n-1}{2}}\right) \tag{8.53}$$

gives a continuous linear map

$$\mathrm{LP}^{-1} : \mathcal{S}(\mathbb{R} \times \mathbb{S}^{n-1}) \longrightarrow S^{-\frac{n}{2}+\frac{1}{2}}(\mathbb{R}^n) \times S^{-\frac{n}{2}-\frac{1}{2}}(\mathbb{R}^n) \tag{8.54}$$

for which LP *is a left inverse.*

PROOF. Since

$$\mathrm{LP} = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}} \begin{pmatrix} D_s^{\frac{n+1}{2}} \cdot R \\ D_s^{\frac{n-1}{2}} \cdot R \end{pmatrix} \tag{8.55}$$

the mapping property (8.52) follows from Lemma 3.9 by duality. Similarly (8.54) follows directly from Lemma 3.9. From (3.113) the composition, $\mathrm{LP} \cdot \mathrm{LP}^{-1}$, is well-defined. Indeed this composite operator is just

$$\mathrm{LP} \cdot \mathrm{LP}^{-1} = \frac{1}{2(2\pi)^{n-1}} \left[D_s^{\frac{n+1}{2}} \cdot R \cdot R^t \cdot D_s^{\frac{n-3}{2}} + D_s^{\frac{n-1}{2}} \cdot R \cdot R^t \cdot D_s^{\frac{n-1}{2}} \right]. \tag{8.56}$$

This corresponds to the splitting by parity

$$f(s, \omega) = f_+(s, \omega) + f_-(s, \omega), \\ f_\pm(s, \omega) = \frac{1}{2} \left[f(s, \omega) \pm (-1)^{\frac{n-1}{2}} f(-s, -\omega) \right], \ f \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1}). \tag{8.57}$$

By Lemma 3.10 the second term in (8.56) annihilates f_- and reproduces f_+. Moreover the proof of Lemma 3.10 applies just as well to show that the first term in (8.56) annihilates f_+ and reproduces f_-. This proves the proposition. □

Returning to (8.51) notice that since it involves integration over a subspace of dimension $n-1$

$$\delta(x\cdot\theta - s' + t) \in \left[S^{-n+2+\gamma+}(\mathbb{R}^n)\right]' \ \forall\ \gamma < 0 \tag{8.58}$$

for each t, s'. Thus

$$\begin{aligned} D_t^{\frac{n-3}{2}}\delta(x\cdot\theta - s' + t) &\in \left[S^{-\frac{n}{2}+\frac{1}{2}+\gamma+}(\mathbb{R}^n)\right]', \\ D_t^{\frac{n-1}{2}}\delta(x\cdot\theta - s' + t) &\in \left[S^{-\frac{n}{2}-\frac{1}{2}+\gamma+}(\mathbb{R}^n)\right]' \ \forall\ \gamma < 0. \end{aligned} \tag{8.59}$$

From (8.52) and (3.113) we can therefore apply LP to u in (8.51).

Lemma 8.28. *There is a unique distribution satisfying* (8.49) *and it satisfies*

$$\begin{gathered} \operatorname{singsupp}(E_{\mathrm{LP}}) \subset \\ \{s' - s + t = 0, \theta = \omega\} \cup \{s' + s + t = 0, \theta = -\omega, |s| \le \rho, |s'| \le \rho\}. \end{gathered} \tag{8.60}$$

Proof. We have already established the existence with

$$E_{\mathrm{LP}}(t, s, \omega; s', \theta) = \mathrm{LP}\begin{pmatrix} u(t, \cdot; s', \theta) \\ D_t u(t, \cdot; s', \theta) \end{pmatrix} \tag{8.61}$$

where u is given by (8.51). Clearly

$$E_{\mathrm{LP}} = \delta(s - s' - t)\delta_\theta(\omega) + \mathrm{LP}\begin{pmatrix} w \\ D_t w \end{pmatrix}. \tag{8.62}$$

Recall that $w = w_+ + w_-$ is conormal. Obviously it suffices to bound the singular support of $Rw_\pm$. This can be written in terms of two projections as

$$\begin{gathered} Rw_\pm = \pi_*\beta^* w_\pm \\ \beta : Z = \{(x, s, \omega); x\cdot\omega = s\} \times \mathbb{R} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n_x \times \mathbb{R} \times \mathbb{S}^{n-1} \\ \pi : Z \longrightarrow \mathbb{R}_s \times \mathbb{S}^{n-1}_\omega \times \mathbb{R} \times \mathbb{S}^{n-1} \end{gathered} \tag{8.63}$$

in which the last two factors are parameters throughout. Now β is transversal to $s' \pm t = x\cdot\theta$, so, by Proposition 5.13, $\beta^* w_\pm$ is conormal to the appropriate hypersurface $\{s' \pm t = x\cdot\theta, x\cdot\omega = s\} \subset Z$. To deduce (8.60) we just apply Corollary 5.2. The surface Z can be parametrized by

$$x = s\omega + \omega',\ \omega'\cdot\omega = 0. \tag{8.64}$$

The fibres of π are just the ω' hyperplanes. Since $h_\pm = s' \pm t - s\theta\cdot\omega + \theta\cdot\omega'$ are defining function for the hypersurfaces and

$$d_{\omega'}h_\pm = \theta - (\theta\cdot\omega)\omega \tag{8.65}$$

we must have $\theta = \pm\omega$ on the critical sets. This gives (8.60).

The uniqueness of the fundamental solution follows by applying LP^{-1}. □

As a consequence of the existence of the fundamental solution for the initial value problem for the transformed equation we can also solve the continuation problem.

PROPOSITION 8.20. *For each* $\theta \in \mathbb{S}^{n-1}$ *there exists a unique distribution*

$$\alpha(t,s,\omega,\theta) \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}), \tag{8.66}$$

satisfying

$$(D_t + D_s)\alpha + V_{\mathrm{LP}}\alpha = 0 \text{ in } \mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1} \tag{8.67}$$

and

$$\alpha(t,s,\omega;\theta) = \delta(s-t)\delta_\theta(\omega) \text{ in } t < -\rho \tag{8.68}$$

where $\rho = \sup\{|x|; x \in \operatorname{supp}(V)\}$.

PROOF. The distribution $H(t)E_{\mathrm{LP}}(t,s,\omega;s',\theta)$ is the forward fundamental solution for the operator $D_t + D_s + L_{\mathrm{LP}}$. The solution to (8.67) and (8.68) is therefore just

$$\alpha(t,s,\omega;s',\theta) = \delta(s-t)\delta_\theta(\omega) + \alpha'(t,s,\omega;s',\theta) \tag{8.69}$$

where

$$\alpha'(t,s,\omega;s',\theta) = \tag{8.70}$$

$$\int_{-\infty}^{t}\int_{\mathbb{S}^{n-1}} E_{\mathrm{LP}}(t-t',s,\omega;s'',\omega')[V_{\mathrm{LP}}\delta(t-s'')\delta_\theta(\omega')]ds''d\omega'. \tag{8.71}$$

□

From (8.47) it follows that

$$\alpha(t,s,\omega;\theta) = \kappa(t-s,\omega;\theta) \text{ in } s > \rho \tag{8.72}$$

where $\kappa \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ is a well-defined distribution. It is called the *scattering kernel*. The way we think of this is as the free wave

$$\alpha_0(t,s,\omega;\theta) = \delta(s-t)\delta_\theta(\omega) \tag{8.73}$$

propagating in from the left and striking the 'potential' which is confined to the region $|s| \le \rho$. Once it has passed through the potential it again freely propagates to the right. Thus the kernel $\kappa(t,\omega;\theta)$ represents the end result of the interaction. We shall find both the regularity and asymptotic properties of κ. At least for n odd, $n \ge 3$, we can do this quite directly. Before doing so we note some more regularity properties of LP .

The finite energy space', $\mathcal{H}(\mathbb{R}^n)$, for the free wave equation is defined to be the completion of $\mathcal{C}^\infty_c(\mathbb{R}^n)\times\mathcal{C}^\infty_c(\mathbb{R}^n)$ with respect to the norm

$$\|(u_0,u_1)\|^2 = \int_{\mathbb{R}^n} |\nabla u_0(x)|^2 dx + \int_{\mathbb{R}^n} |u_1(x)|^2 dx. \tag{8.74}$$

PROPOSITION 8.21. *The inclusion of* $\mathcal{C}^\infty_c(\mathbb{R}^n)\times\mathcal{C}^\infty_c(\mathbb{R}^n)$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *extends by continuity to an injection (inclusion) from* $\mathcal{H}(\mathbb{R}^n)$, *the wave group* $U_V(t)$ *extends to a strongly continuous group of bounded operators on* $\mathcal{H}(\mathbb{R}^n)$, LP *extends by continuity to an isometric isomorphism*

$$\mathrm{LP} : \mathcal{H}(\mathbb{R}^n) \longleftrightarrow L^2(\mathbb{R}\times\mathbb{S}^{n-1}) \tag{8.75}$$

and $W_V(t) = \mathrm{LP}\cdot U_V(t)\cdot \mathrm{LP}^{-1}$ *is a strongly continuous group of bounded operators on* $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$.

PROOF. We can start by proving (8.75). From Lemma 3.10 the second term in (8.55) is an isometric isomorphism onto the subspace of $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ with parity as in (3.119). Applying this to the first part of (8.55) we see that for any $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}\frac{1}{2(2\pi)^{n-1}}\int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} D_s^{\frac{n+1}{2}}Ru_0\overline{D_s^{\frac{n+1}{2}}Ru_0}dsd\omega \\ &= \frac{1}{2(2\pi)^{n-1}}\int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} D_s^{\frac{n-1}{2}}Ru_0\overline{D_s^{\frac{n-1}{2}}R\Delta u_0}dsd\omega \\ &= \int\limits_{\mathbb{R}^n} u_0\overline{\Delta u_0}dx \\ &= \int\limits_{\mathbb{R}^n}|\nabla u_0(x)|^2dx.\end{aligned} \tag{8.76}$$

This shows that LP extends to a linear isometric map from $\mathcal{H}(\mathbb{R}^n)$ to a closed subspace of $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$. In fact it only remains to show that the first term in (8.55) is surjective onto the subspace with the opposite parity to that in (3.120). This follows directly from the injectivity of LP^{-1} shown in Proposition 8.19. Thus we have proved (8.75).

Now the fact that $\mathcal{H}(\mathbb{R}^n)$ injects into $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ follows from (8.75). Certainly the second factor in $\mathcal{H}(\mathbb{R}^n)$ is just $L^2(\mathbb{R}^n)$. The first is not quite $H^1(\mathbb{R}^n)$ since the norm on u_0 in (8.74) does not include the L^2 norm. However, from (8.75) it follows that

$$u_0(x) = 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}}R^tD_s^{\frac{n-1}{2}}k',\ D_sk' = LP(u_0,u_1)\in L^2(\mathbb{R}\times\mathbb{S}^{n-1}). \tag{8.77}$$

Thus k' is locally in $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ and since the value of u_0 in $|x|\le R$ only depends on k' in $|s|\le R$ it follows from Lemma 3.10 that $u_0\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as claimed.

Next we want to show that $U_V(t)$ extends to a group of operators on $\mathcal{H}(\mathbb{R}^n)$. Consider first the case $V\equiv 0$. For smooth initial data of compact support set

$$E(t) = \int\limits_{\mathbb{R}^n}|\nabla u(t,x)|^2+|D_tu(t,x)|^2dx. \tag{8.78}$$

This is certainly a $\mathcal{C}^\infty$ function and by integration by parts

$$\frac{dE(t)}{dt} = \int\limits_{\mathbb{R}^n}[\nabla u\cdot\overline{\partial_t\nabla u}+\partial_t\nabla u\cdot\overline{\nabla u}+\partial_t^2u\overline{\partial_t u}+\partial_tu\overline{\partial_t^2u}]dx = 0. \tag{8.79}$$

Thus, for $V\equiv 0$, $U_0(t)$ extends to a unitary group of operators on $\mathcal{H}(\mathbb{R}^n)$. For a general $V\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ we get in place of (8.79)

$$\frac{dE(t)}{dt} = \int\limits_{\mathbb{R}^n}[-Vu\overline{\partial_tu}-\partial_tu\overline{Vu}]dx. \tag{8.80}$$

Since V has compact support the inclusion of $\mathcal{H}(\mathbb{R}^n)$ into $L^2_{loc}(\mathbb{R}^n)$ means that

$$|\int\limits_{\mathbb{R}^n}\partial_tu(t,x)\overline{V(x)u(t,x)}dx|\le CE(t). \tag{8.81}$$

Thus for some constant $c>0$ $e^{ct}E(t)$ is decreasing. It follows that $U_V(t)$ is always a bounded group of operators. The strong continuity is immediate as well.

From (8.75) the statement concerning the transformed group, $W_V(t)$, follows. This completes the proof of the proposition. □

Of course if V is real-valued then the free energy (8.74) can be replaced by the perturbed energy

$$\|(u_0,u_1)\|_V^2 = \int_{\mathbb{R}^n} |\nabla u_0|^2 + |u_1|^2 + V(x)|u_0|^2 dx. \tag{8.82}$$

In general this need not be positive-definite so it does not lead immediately to a Hilbert space. If it is positive-definite (for example if $V \geq 0$) then $U_V(t)$ is unitary on the completion, $\mathcal{H}_V(\mathbb{R}^n)$, of $\mathcal{C}_c^\infty(\mathbb{R}^n)\times\mathcal{C}_c^\infty(\mathbb{R}^n)$ with respect to this norm. In this case it follows from Proposition 8.21 that $\mathcal{H}_V(\mathbb{R}^n)$ is isomorphic to $\mathcal{H}_0(\mathbb{R}^n)$ an the norms are equivalent. Even if the bilinear form (8.82) is not positive definite it does extend by continuity to a bilinear form on $\mathcal{H}(\mathbb{R}^n)$. As will be shown in Chapter 11 the negative part corresponds to true eigenvalues of the operator $\Delta + V$. In case V has complex values this construction does not make sense, so we shall not exploit it.

Consider further the group of operators, $W_V(t)$, conjugate to the wave group. The Schwartz kernel of $W_V(t)$ is $E_{\mathrm{LP}}(t,s,\omega;s',\theta)$. We have obtained information on the singular support of E_{LP} in Lemma 8.28 and we can also easily find some bounds on the support of this distribution.

Lemma 8.29. *If* $\operatorname{supp}(V) \subset \{|x| \leq \rho\}$ *then the Schwartz' kernel* E_{LP} *of the group* $W_V(t)$ *satisfies*

$$\begin{gathered} E_{\mathrm{LP}}(t,s,\omega;s'\theta) = \delta(t-s+s')\delta_\theta(\omega) \text{ in} \\ \{s < -\rho + t_- + (s'-\rho)_+\} \cup \{s > \rho + t_+ - (s'+\rho)_-\} \end{gathered} \tag{8.83}$$

where $X_+ = H(X)X$ *and* $X_- = H(-X)X$.

Proof. For each s' there is a unique solution to (8.49). Moreover, by Lemma 8.27, the solution satisfies the free equation in $|s| > \rho$. Thus we must have

$$E_{\mathrm{LP}}(t,s,\omega;s',\theta) = \begin{cases} \gamma_-(t-s,\omega;s'\theta) \text{ in } s<\rho \\ \gamma_+(t-s,\omega;s'\theta) \text{ in } s>\rho. \end{cases} \tag{8.84}$$

Now, the initial condition in (8.49) implies

$$\begin{cases} \gamma_-(-s,\omega;s'\theta) = \delta(s-s')\delta_\theta(\omega) \text{ in } s<-\rho \\ \gamma_+(-s,\omega;s',\theta) = \delta(s-s')\delta_\theta(\omega) \text{ in } s>\rho. \end{cases} \tag{8.85}$$

This proves the equality in (8.83) in the region $\{s < -\rho + H(-t)t$. In particular this gives half of the first part of (8.83), corresponding to $s' \leq \rho$. If $s' > \rho$ then it also shows that $E_{\mathrm{LP}} = 0$ in $s < -\rho + H(-t)t + (s'-\rho)$. This gives the first part of (8.83), the second part follows similarly or by reflecting in t, s and s'. □

CHAPTER 9

Conormal distributions at a submanifold

We have already made quite extensive use of conormal distributions associated to hypersurfaces and along the way similar distributions related to submanifolds of higher codimension have appeared. The simplest of these is $\delta(x)$, the Dirac distribution associated to the point submanifold $\{0\} \subset \mathbb{R}^n$. It is high time that we defined these distributions in general and examined their basic properties. The generalization from hypersurfaces is particularly important since it leads directly to the theory of pseudodifferential operators, a fundamental component of 'microlocal analysis'.

Initially the analysis below is directly parallel to the hypersurface case examined in Chapter 3. That is, we first define the distributions by iterative regularity in Sobolev spaces and then introduce the symbol by using the Fourier transform. Since this is done in local coordinates we need to check carefully the coordinate independence of the space of conormal distributions of a fixed order.

To treat push-forward and pull-back operations on these general conormal distributions we use an alternative characterization in terms of the push-forward, under an appropriate fibration, of conormal distributions associated to a hypersurface. This is in line with our generally geometric approach. The main advantage of switching to this second characterization is that it is the one we adopt in the more general case of Lagrangian distributions in Chapter 17.

A submanifold $X \subset \mathbb{R}^n$ is a subset with the property that for each $\bar{x} \in X$ there is a diffeomorphism of open subsets of $\mathbb{R}^n$, $F : \Omega \longleftrightarrow \Omega', \bar{x} \in \Omega, 0 \in \Omega'$ such that

$$F(X \cap \Omega) = M \cap \Omega' \tag{9.1}$$

where $M \subset \mathbb{R}^n$ is a linear space. Since we can subject M to a further linear transformations we may as well take as our model spaces

$$M = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_1 = \cdots = x_k = 0\}. \tag{9.2}$$

Thus M is of dimension $q = n - k$ and codimension k in $\mathbb{R}^n$. We shall generally write the coordinates as

$$y_i = x_i, i = 1, \dots, k, \quad z_j = x_{j+k}, j = 1, \dots, q. \tag{9.3}$$

Thus $x = (y, z), y \in \mathbb{R}^k, z \in \mathbb{R}^q \cong M$.

Consider the space of all $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$ which are tangent to M :

$$V \in \mathcal{V}(M) \Longleftrightarrow V = \sum_{i=1}^{k} a_i D_{y_i} + \sum_{j=1}^{q} b_j D_{zj}, \quad a_i, b_j \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{9.4}$$

such that $V y_i = 0$ on M for $i = 1, \dots, k$. Clearly the vector fields

$$A_{i\ell} = y_i D_{y_\ell}, B_j = D_{z_j}, \quad i, \ell = 1, \dots k,\ j = 1, \dots, q \tag{9.5}$$

are all in $\mathcal{V}(M)$.

LEMMA 9.30. *$\mathcal{V}(M)$ is a left $\mathcal{C}^\infty(\mathbb{R}^n)$-module and as such is generated by the elements* (9.5)*.*

PROOF. Directly from the definition, (9.4), $\mathcal{V}(M)$ is a $\mathcal{C}^\infty$-module. If $V \in \mathcal{V}(M)$ then from (9.4) $Vy_i = a_i = 0$ on M if and only if $a_i = \sum_{\ell=1}^{k} a_{i\ell}y_\ell$ with $a_{i\ell} \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus any $V \in \mathcal{V}(M)$ can be written

$$V = \sum_{i,\ell=1}^{k} a_{i\ell}y_\ell D_{y_i} + \sum_{j=1}^{q} b_j D_{z_j} \tag{9.6}$$

proving the lemma. □

The spaces of conormal distributions with respect to M are determined by the stability of their regularity under the application of elements of $\mathcal{V}(M)$.

DEFINITION 9.8. If $s \in \mathbb{R}$ and M is given by (9.2) then $I_c^{(s)}(\mathbb{R}^n, M) \subset \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ consists of those distributions of compact support, u, for which

$$V_1 \dots V_N u \in H^s(\mathbb{R}^n) \quad \forall\ V_1, \dots, V_N \in \mathcal{V}(M) \text{ and all } N \in \mathbb{N}. \tag{9.7}$$

Certainly we have

$$\begin{gathered} I_c^{(s)}(\mathbb{R}^n, M) \subseteq I_c^{(s')}(\mathbb{R}^n, M) \quad \text{ if } s \ge s' \\ I_c^{(\infty)}(\mathbb{R}^n, M) = \bigcap_s I_c^{(s)}(\mathbb{R}^n, M) = \mathcal{C}_c^\infty(\mathbb{R}^n). \end{gathered} \tag{9.8}$$

Let us define the spaces, $\operatorname{Diff}_M^N(\mathbb{R}^n)$, of $\mathcal{V}(M)$-differential operators of order N, inductively by setting

$$\begin{gathered} \operatorname{Diff}_M^0(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n) \\ \operatorname{Diff}_M^1(\mathbb{R}^n) = \mathcal{V}(M) + \mathcal{C}^\infty(\mathbb{R}^n) \\ \operatorname{Diff}_M^N(\mathbb{R}^n) = \mathcal{V}(M) \cdot \operatorname{Diff}_M^{N-1}(\mathbb{R}^n) + \operatorname{Diff}_M^{N-1}(\mathbb{R}^n), \end{gathered} \tag{9.9}$$

as operators on $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$. Thus the union over N of these spaces of operators is the enveloping algebra of $\mathcal{V}(M)$. Then (9.7) can be written

$$\operatorname{Diff}_M^N(\mathbb{R}^n)u \subset H^s(\mathbb{R}^n) \quad \forall\ N. \tag{9.10}$$

LEMMA 9.31. *Each $\operatorname{Diff}_M^N(\mathbb{R}^n)$ is a 2-sided $\mathcal{C}^\infty(\mathbb{R}^n)$-module, invariant under the taking of transposes with respect to any non-vanishing $\mathcal{C}^\infty$ measure and spanned by*

$$y^\alpha D_y^\beta D_z^\gamma \quad |\alpha| = |\beta|,\ |\alpha| + |\gamma| \le N. \tag{9.11}$$

PROOF. All these statements follow easily by induction on N, for example that $\operatorname{Diff}_M^N(\mathbb{R}^n)$ is a $\mathcal{C}^\infty(\mathbb{R}^n)$-module follows directly from (9.11), as does the invariance under transposes since

$$V^t = -V + a,\ a \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{9.12}$$

To prove that $\operatorname{Diff}_M^N(\mathbb{R}^n)$ is spanned by (9.11) we can suppose this true for $N = N' - 1$. Then it is only necessary to note the simple commutation identities

$$\begin{gathered} D_{z_j} y^\alpha D_y^\beta D_z^\gamma = y^\alpha D_y^\beta (D_{z_j} D_z^\gamma), \\ y_i D_{y_l} y^\alpha D_y^\beta D_z^\gamma = (y_i y^\alpha)(D_{y_l} D_y^\beta) D_z^\beta - i\alpha_l (y_i y^{\alpha'}) D_y^\beta D_z^\gamma \end{gathered} \tag{9.13}$$

where $\alpha'_r = \alpha_r$ except for $r = i, \alpha'_i = \alpha_i - 1$. □

The invariance under transpose applied to (9.11), and (9.11) directly, show that Definition 9.8 is equivalent to either

$$y^\alpha D_y^\beta D_z^\gamma u \in H_c^s(\mathbb{R}^n) \quad \forall\ |\alpha| = |\beta| \text{ or}$$
$$D_y^\beta D_z^\gamma y^\alpha u \in H_c^s(\mathbb{R}^n) \quad \forall\ |\alpha| = |\beta|. \tag{9.14}$$

From the first of these it follows that

$$\text{singsupp}(u) \subset M \quad \forall\ u \in I_c^{(s)}(\mathbb{R}^n, M). \tag{9.15}$$

The second can also be written in the suggestive form

$$D_z^\gamma y^\alpha u \in H_c^{s+|\alpha|}(\mathbb{R}^n) \quad \forall\ \alpha, \gamma. \tag{9.16}$$

Thus if $u \in I_c^{(s)}(\mathbb{R}^n, M)$ is multiplied by a $\mathcal{C}^\infty$ function, f, vanishing on M with its first p-derivatives then

$$D^\alpha f|_M = 0\ \forall\ |\alpha| \le p \Longrightarrow fu \in I_c^{(s+p)}(\mathbb{R}^n, M) \subset H^{s+p}(\mathbb{R}^n). \tag{9.17}$$

Lemma 9.32. *Suppose $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ such that*

$$F(\Omega_1 \cap M) = \Omega_2 \cap M, \tag{9.18}$$

with M given by (9.2)*, then*

$$F^*[I_c^{(s)}(\mathbb{R}^n, M) \cap \mathcal{C}_c^{-\infty}(\Omega_2)] = I_c^{(s)}(\mathbb{R}^n, M) \cap \mathcal{C}_c^{-\infty}(\Omega_1). \tag{9.19}$$

Proof. If u has compact support in Ω then $u \in I_c^{(s)}(\mathbb{R}^n, M)$ if $\mathcal{V}(M)^p u \subset H_c^s(\Omega)$ for every p. Since both the Lie algebra $\mathcal{V}(M)$ and the Sobolev spaces are invariant under F the invariance (9.19) follows. □

A characterization of $I_c^{(s)}(\mathbb{R}^n, M)$ in terms of symbols is given by use of the partial Fourier transform across the submanifold:

$$\tilde{u}(z, \zeta) = \int e^{-iy\cdot\zeta} u(y, z) dz. \tag{9.20}$$

Proposition 9.22. *If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and M is given by* (9.2) *then $u \in I_c^{(s)}(\mathbb{R}^n, M)$ if and only if the partial Fourier transform, defined by* (9.20)*, satisfies*

$$\left(1 + |\zeta|^2\right)^{\frac{s+|\beta|}{2}} D_z^\beta D_\zeta^\alpha \tilde{u}(z, \zeta) \in \mathcal{C}_c^\infty(\mathbb{R}_z^q; L^2(\mathbb{R}_\zeta^k)), \quad \forall \alpha, \beta. \tag{9.21}$$

Proof. Taking the full Fourier transform

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx, \tag{9.22}$$

where $\xi = (\zeta, \eta) \in \mathbb{R}^k \times \mathbb{R}^q$, gives a $\mathcal{C}^\infty$ function $\hat{u}$. The second form of (9.14) shows that $u \in I_c^{(s)}(\mathbb{R}^n, M)$ if and only if

$$\zeta^\gamma \eta^\beta D_\eta^\alpha \hat{u} \in (1 + |\eta|^2 + |\zeta|^2)^{-\frac{s}{2}} L^2(\mathbb{R}^n) \quad \forall\ |\alpha| = |\beta|, \gamma. \tag{9.23}$$

Since (9.23) holds for all β it is certainly equivalent to

$$D_\eta^\alpha \hat{u}(\eta, \xi) \in (1 + |\zeta|^2)^{-N} \left(1 + |\eta|^2\right)^{-\frac{s+|\alpha|}{2}} L^2(\mathbb{R}^n) \forall\ N, \alpha. \tag{9.24}$$

Taking the inverse Fourier transform from ζ to z gives (9.21). □

As in the hypersurface case we can relate (9.21) to more conventional 'L^∞-based' symbol spaces. The latter are defined by extension from (3.41).

$$
\begin{aligned}
&a \in S^m(\mathbb{R}^q, \mathbb{R}^k) \Longleftrightarrow a \in \mathcal{C}^\infty(\mathbb{R}^{q+k}) \text{ and} \\
&\sup_{K\times\mathbb{R}^q} (1+|\eta|)^{-m+|\alpha|} |D_\eta^\alpha D_z^\beta a(z,\eta)| < \infty \\
&\qquad \forall\ K \subset\subset \mathbb{R}^n, \alpha \in \mathbb{N}^k \text{ and } \beta \in \mathbb{N}^q.
\end{aligned} \tag{9.25}
$$

A simple example of such a symbol is a $\mathcal{C}^\infty$ function which is homogeneous of degree m in η in $|\eta| \geq 1$. One of the main reasons that the L^∞ bounds (9.25) are preferable to the L^2 estimates (9.21) is the multiplicative properties:

$$
S^m(\mathbb{R}^q, \mathbb{R}^k) \cdot S^{m'}(\mathbb{R}^q, \mathbb{R}^k) \subset S^{m+m'}(\mathbb{R}^q, \mathbb{R}^k) \tag{9.26}
$$

as follows easily from Leibniz' formula. Similarly if $\tilde{u}$ satisfies (9.21) and $\phi \in S^0(\mathbb{R}^q, \mathbb{R}^k)$ then $\phi\tilde{u}$ satisfies the same estimates. This allows us to localize the estimates to cones.

Suppose that $\phi \in \mathcal{C}^\infty(\mathbb{R}^k)$ is supported in

$$
\{|\eta| \geq \frac{1}{2}\} \cap \{|\eta| \leq \pm 2\eta_j\} \tag{9.27}
$$

By relabelling and reflecting the coordinates we might as well take $j = 1$ with the plus sign. Assume further that ϕ is homogeneous of degree 0 in $|\eta| \geq 1$, so is in $S^0(\mathbb{R}^q, \mathbb{R}^k)$. On the support of ϕ we can introduce the projective coordinates

$$
z, \eta_1, t_j = \eta_j / \eta_1 \quad j = 2, \dots, k. \tag{9.28}
$$

Now if $\tilde{u}$ satisfies (9.21) then so does $\phi\tilde{u}$. In terms of the coordinates (9.28) these estimates can be written

$$
\eta_1^{s+p} D_t^{\beta'} D_z^\alpha D_{\eta_1}^p \phi\tilde{u} \in \eta_1^{-\frac{k-1}{2}} L^2(\mathbb{R}^n_{z,t,\eta_1}). \tag{9.29}
$$

where the extra factors of η_1 come from the change of measure $d\eta = \eta_1^{k-1} dt d\eta_1$. We have written this in a form emphasizing that the new variables, t, behave in exactly the same way as the 'parameters' z. Thus as before we easily conclude that

$$
\eta_1^{s+p+\frac{k-1}{2}} D_{(t,z)}^\gamma D_{\eta_1}^p \phi\tilde{u}(z, \eta_1, \eta, t) \in \mathcal{C}^\infty(\mathbb{R}^{q+k-1}, L^2(\mathbb{R})). \tag{9.30}
$$

Now we can apply the one-dimensional result, that (3.29) implies (3.27) for $M > -s - \frac{1}{2}$, to conclude from (9.30) that

$$
\phi\tilde{u}(z, \eta_1, \eta, t) \in S^m(\mathbb{R}^{q+k-1}, \mathbb{R}),\ m > -s - \frac{k}{2}. \tag{9.31}
$$

This is the main part of the proof of:

Lemma 9.33. *If $u \in I_c^{(s)}(\mathbb{R}^n, M)$, with M given by* (9.2)*, then $\tilde{u}$ defined by* (9.20) *is an element of $S^m(\mathbb{R}^q, \mathbb{R}^k)$ for $m > -s - \frac{k}{2}$ and conversely if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $\tilde{u} \in S^m(\mathbb{R}^q, \mathbb{R}^k)$ then $u \in I_c^{(s)}(\mathbb{R}^m, M)$ for any $s < -m - \frac{k}{2}$.*

Proof. From (9.31) it follows that if ϕ is a conic cut-off supported in (9.27) for $j = 1$ then $\phi\tilde{u} \in S^m(\mathbb{R}^q, \mathbb{R}^k)$. The same argument applies for other j so ϕ can be chosen to form a partition of unity, in $S^0(\mathbb{R}^q, \mathbb{R}^k)$, subordinate to the cones (9.27). Since $\tilde{u}$ is $\mathcal{C}^\infty$ it follows that $\tilde{u} \in S^m(\mathbb{R}^q, \mathbb{R}^k)$.

The converse is even easier If $\tilde{u} \in S^m(\mathbb{R}^q, \mathbb{R}^k)$ with $m < -s - \frac{k}{2}$ then

$$
\left|(1+|\eta|^2)^{\frac{s}{2}} D_z^\alpha \tilde{u}(\eta, z)\right| \leq C(1+|\eta|)^{m+s} \in L^2(\mathbb{R}^k) \quad \forall\ z \in \mathbb{R}^q. \tag{9.32}
$$

Thus $u \in H^s_c(\mathbb{R}^n)$. The higher order estimates in (9.21) follow similarly. □

As for hypersurfaces we think of the order of a conormal distribution as being related to the power law, m, in the estimates (9.25) on its partial Fourier transform, rather than to Sobolev regularity. The precise order has the usual, somewhat obscure looking, normalizing terms; the virtue in suffering with these will eventually be rewarded in simplification (or at least consistency) in the treatment of Lagrangian distribution (should the reader last that long!)

Definition 9.9. If M is given by (9.2) then for any $m \in \mathbb{R}$ we set

$$I^m_c(\mathbb{R}^n, M) = \{u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n); \tilde{u}(z,\eta) \in S^{m+\frac{n}{4}-\frac{k}{2}}(\mathbb{R}^q, \mathbb{R}^k)\}, \tag{9.33}$$

where $q = n-k$, the partial Fourier transform is defined by (9.20) and the symbol spaces are given by (9.25).

Notice that (9.33) is consistent with Definition 3.3, for a hypersurface. The filtration by order goes in the opposite direction to that by Sobolev regularity

$$I^m_c(\mathbb{R}^n, M) \subset I^{m'}_c(\mathbb{R}^n, M) \quad \text{if } m' \geq m. \tag{9.34}$$

As a direct consequence of Lemma 9.33 we find

$$\begin{aligned} I_c^{-s-\frac{n}{4}-\epsilon}(\mathbb{R}^n, M) \subset I_c^{(s)}(\mathbb{R}^n, M) \subset I_c^{-s-\frac{n}{4}+\epsilon}(\mathbb{R}^n, M) \\ I_c^{(-m-\frac{n}{4}+\epsilon)}(\mathbb{R}^n, M) \subset I_c^{m}(\mathbb{R}^n, M) \subset I_c^{(-m-\frac{n}{4}-\epsilon)}(\mathbb{R}^n, M) \end{aligned} \ \forall\ \epsilon > 0. \tag{9.35}$$

From this it follows easily that each $I^m_c(\mathbb{R}^n, M)$ is a $\mathcal{C}^\infty$-module, in fact more is true:

Lemma 9.34. *For any N all $m \in \mathbb{R}$*

$$\operatorname{Diff}^N_M(\mathbb{R}^n) \cdot I^m_c(\mathbb{R}^n, M) \subset I^m_c(\mathbb{R}^n, M) \tag{9.36}$$

and if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$

$$f = 0 \text{ to order } p \text{ on } M \Longrightarrow f \cdot I^m_c(\mathbb{R}^n, M) \subset I^{m-p}_c(\mathbb{R}^n, M). \tag{9.37}$$

Proof. To check that $I^m_c(\mathbb{R}^n, M)$ is a $\mathcal{C}^\infty$-module we can use Taylor's formula to write

$$\phi(y,z) = \phi_0(z) + \sum_{i=1}^k y_i \psi_i(\eta, z), \quad \psi_i \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{9.38}$$

Directly from the definition

$$\widetilde{\phi_0 u}(z,\eta) = \phi_0(z)\tilde{u}(z,\eta) \in S^{m+\frac{n}{4}-\frac{k}{2}}(\mathbb{R}^q, \mathbb{R}^k) \tag{9.39}$$

if $u \in I^m_c(\mathbb{R}^n, M)$, so $\phi_0 u \in I^m_c(\mathbb{R}^n, M)$. On the other hand we know that (9.17) holds, so applying (9.35) twice

$$y_i \psi_i u \in I^{m-1+\epsilon}_c(\mathbb{R}^n, M) \quad \forall \epsilon > 0. \tag{9.40}$$

Thus $\phi u \in I^m_c(\mathbb{R}^n, M)$. Having shown this, (9.40) must hold with $\epsilon = 0$ since

$$\widetilde{y^\alpha u} = (-1)^{|\alpha|} D^\alpha_\eta \tilde{u}(z,\eta) \in S^{m-|\alpha|+\frac{n}{4}-\frac{k}{2}}(\mathbb{R}^q, \mathbb{R}^k). \tag{9.41}$$

Similar arguments give (9.36) and (9.37). □

As in Proposition 3.11 the 'integrated' version of (9.36) for $N = 1$:

$$\mathcal{V}(M) \cdot I_c^m(\mathbb{R}^n, M) \subset I_c^m(\mathbb{R}^n, M) \tag{9.42}$$

is the analogue of (9.19), i.e.

$$F^*[I_c^m(\mathbb{R}^n, M) \cap \mathcal{C}_c^{-\infty}(\Omega_2)] = I_c^m(\mathbb{R}^n, M) \cap \mathcal{C}_c^{-\infty}(\Omega_1) \tag{9.43}$$

under the hypotheses made on F in Lemma 9.32.

To prove (9.43) we proceed as in the proof of (3.54) by factorizing F, in the form

$$F = F_0 \cdot F_1 \cdot F_{(2)} \tag{9.44}$$

where F_0 is a diffeomorphism of $\Omega_1 \cap M$ to $\Omega_2 \cap M$ in $\mathbb{R}^q$ lifted to $\mathbb{R}^n$

$$F_0(y, z) = (y, F_0'(z)) \tag{9.45}$$

as is the linear part

$$F_1(y, \eta) = (G(z)y, z), \ G : \Omega_1 \cap M \longrightarrow GL(k, \mathbb{R}). \tag{9.46}$$

The 'remainder term' $F_{(2)}$ leaves M fixed to second order

$$F_{(2)}(y, z) = (y + 0(|y|^2), z + 0(|y|)) \text{ as } |y| \longrightarrow 0. \tag{9.47}$$

To construct such a factorization, define F_0' by noting that $F(0, z) = (0, F_0'(z))$. Then F_1 is just the linear part at $y = 0$ of $F_0^{-1} \circ F$, which is clearly of the form

$$F_0^{-1} \cdot F(y, z) = (G(z)y + 0(|y|^2), z + 0(|y|)) \tag{9.48}$$

from which (9.46) and (9.47) follow. Notice that $F_{(2)}$ and F_1 can be viewed as maps between small neighbourhoods of $\Omega_1 \cap M$ and F_0 gives a map from such a neighbourhood to a neighbourhood of $\Omega_2 \cap M$. In view of (9.15) we can assume that $u \in I_c^m(\mathbb{R}^n, M)$ has support in any preassigned neighbourhood of $M \cap \Omega_2$, if $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$, so we can act as though the diffeomorphism is defined globally on $\mathbb{R}^n$.

The invariance of $I_c^k(\mathbb{R}^n, M)$ under F_0 is trivial, since

$$\widetilde{F_0^* u}(z, \eta) = \tilde{u}(F_0'(z), \eta). \tag{9.49}$$

Similarly the invariance under F_1 follows by a change of integral variables, giving

$$\begin{aligned} \widetilde{F_1^* u}(z, \eta) &= \int e^{-iy\cdot\eta} u(G(z)y, z) dy \\ &= \tilde{u}(z, (G^t)^{-1}(z)\eta) |\det G(z)|. \end{aligned} \tag{9.50}$$

The invariance under $F_{(2)}$ follows by an homotopy argument. From (9.47) the one-parameter family of maps

$$F_{(2),r}(y, z) = (1 - r)(y, z) + rF_{(2)}(y, z) \quad r \in [0, 1] \tag{9.51}$$

(using the additive structure of $\mathbb{R}^n$) are all diffeomorphisms near $M \cap \Omega_1$. Thus it suffices to show that

$$\frac{d}{dr} F_{(2),r}^* u \in I_c^m(\mathbb{R}^n, M) \quad \forall \ u \in I_c^m(\mathbb{R}^n, M). \tag{9.52}$$

As in (3.72) this follows from the fact that

$$\frac{d}{dr} F_{(2),r}^* u = \sum_{i=1}^k y_i V_i \cdot F_{(2),r}^* u, \ V_i \in \mathcal{V}(M), \tag{9.53}$$

using (9.35).

This proves (9.43). We record the behaviour of the partial Fourier transform.

Proposition 9.23. *For any diffeomorphism, as in Lemma 9.32,* (9.43) *holds and if* $F^{\#}$ *is defined by*

$$(F^{\#}\cdot a)(z,\eta)=a(F_0(z),(G^t)^{-1}(z)\eta)|\det G(z)| \tag{9.54}$$

where $F(0,z)=(0,F_0(z))$ *and* $G_{ij}(z)=\frac{\partial F_j}{\partial y_i}(0,z)$, $i,j=1,\ldots,k$ *then for each* $u\in I_c^m(\mathbb{R}^n,M)$, *with* M *given by* (9.2),

$$\widetilde{F^*u}(z,\eta)\equiv F^{\#}\cdot\tilde{u}\mod S^{m+\frac{n}{4}-\frac{k}{2}-1}(\mathbb{R}^q,\mathbb{R}^k). \tag{9.55}$$

Proof. Combining (9.49) and (9.50) this is just a matter of showing that

$$F^*_{(2)}u-u\in I_c^{m-1}(\mathbb{R}^n,M)\quad\forall\, u\in I_c^m(\mathbb{R}^n,M). \tag{9.56}$$

This in turn follows from (9.43), (9.34) and (9.37). □

Now Proposition 9.23 shows us how to define the symbol of these conormal distributions. We set

$$\sigma_m(u)=\tilde{u}(z,\eta)|d\eta|\qquad\mod S^{m+\frac{n}{4}-\frac{k}{2}-1}(\mathbb{R}^q,\mathbb{R}^k)\cdot|d\eta|. \tag{9.57}$$

The symbol is, just as in the hypersurface case, an equivalence class of products of symbols and fibre densities. The reason for including the density term is, as usual, to absorb the Jacobian factor in (9.54).

Recall that for a general submanifold $X\subset\mathbb{R}^n$ the conormal fibre to X, in $\mathbb{R}^n$, at $\bar{x}$ is a quotient of the ideal

$$\mathcal{I}(X,\mathbb{R}^n)=\{f\in\mathcal{C}^\infty(\mathbb{R}^n);f=0\text{ on }X\}. \tag{9.58}$$

If $\mathcal{I}(\{\bar{x}\},\mathbb{R}^n)=[f\in\mathcal{C}^\infty(\mathbb{R}^n);f(\bar{x})=0]$ then

$$N^*_{\bar{x}}X=\mathcal{I}(X,\mathbb{R}^n)/\mathcal{I}(\{X\},\mathbb{R}^n)\cdot\mathcal{I}(X,\mathbb{R}^n). \tag{9.59}$$

This is always a vector space of dimension equal to the codimension of X.

If $y_1,\ldots,y_k$ are defining functions for X near $\bar{x}$ and $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ is identically one in some neighbourhood of $\bar{x}$ and supported in a sufficiently small neighbourhood of $\bar{x}$ then $f\in\mathcal{I}(X,\mathbb{R}^n)$ can be written

$$f=\sum_{i=1}^k\psi_iy_i\phi+f',\psi_i\in\mathcal{C}^\infty(\mathbb{R}^n),f'\in\mathcal{I}(\bar{x},\mathbb{R}^n)\cdot\mathcal{I}(X,\mathbb{R}^n). \tag{9.60}$$

Thus the elements $y_i\phi$ give a basis for $N^*_{\bar{x}}X$, usually written $dy_i,i=1,\ldots,k$ (being independent of ϕ.) Thus

$$N^*_{\bar{x}}\ni[f]=\sum_{i=1}^n\eta_idy_i\quad\eta_i=\frac{df}{\partial y_i}(\bar{x}) \tag{9.61}$$

in any local coordinates $y_1,\ldots,y_k,z_1,\ldots,z_q$ near $\bar{x}$. The conormal bundle to X

$$N^*X=\bigcup_{\bar{x}\in X}N^*_{\bar{x}}X \tag{9.62}$$

is a smooth submanifold of $\mathbb{R}^n\times\mathbb{R}^n$.

Of course (9.61) means that local coordinates (y,z) in which X is locally given by $y=0$ induce coordinates in N^*X, (z,η).

LEMMA 9.35. *If (z, η) are interpreted as local coordinates in N^*M, where M is given by* (9.2)*, then $\sigma_m(u)$ defined by* (9.57) *is an equivalence class of densities on the fibres of N^*M defined independently of the local coordinates.*

PROOF. If F_0 is a local diffeomorphism as in (9.45) then it does not alter the dual coordinate η, so just acts as a coordinate transformation on M, consistent with (9.49).

If F_1 is of the form (9.46) then it alters the choice of defining functions from y_i to

$$y_l' = \sum_{l=k}^{k} G_{i_l}(z) y_l \quad i = 1, \dots, k. \tag{9.63}$$

On the dual coordinates this induces the transformation

$$\eta_l' = \sum_{i=1}^{k} (G^{-1}(z))_{il} \eta_i. \tag{9.64}$$

Thus, $|d\eta| = |\det G(z)||d\eta'|$ so (9.50) shows that $\sigma_m(u)$ defined by (9.57) in the two coordinate systems gives the same density.

Under a general coordinate transformation the invariance now follows from (9.54) and the decomposition (9.44). □

Since we have made all our constructions coordinate independent it is now a simple matter to transfer Definition 9.9 to a general submanifold $M \subset X$. We shall not assume our submanifolds to be closed so we must introduce some condition to ensure that the conormal distributions do not misbehave near the boundary of M.

DEFINITION 9.10. If $M \subset X$ is a smooth submanifold of codimension k then $I^m(X, M)$ consists of those distributions $u \in \mathcal{C}^{-\infty}(X)$ with

$$\text{singsupp}(u) \subset M \tag{9.65}$$

and such that for each point $p \in M$ there are local coordinates based at p in terms of which M is given by (9.2) and if $\phi \in \mathcal{C}^\infty(X)$ has compact support in the coordinate patch then $\phi u \in I^m(\mathbb{R}^n, M_0)$, as in (9.33).

The coordinate invariance proved in the case of the model submanifold, (9.2), means that if $u \in I^m(X, M)$ then $\phi u \in I^m(\mathbb{R}^n, M_0)$ whenver $\phi \in \mathcal{C}^\infty(X)$ has support in a coordinate patch in terms of which $M = M_0$. It follows in particular that if $F : X \longrightarrow X$ is a diffeomorphism then

$$F^* : I^m(X, F(M)) \longrightarrow I^m(X, M) \tag{9.66}$$

for any submanifold M. Similarly the invariance of the symbol map means that there is a well-defined symbol map

$$\sigma_m : I^m(X, M) \longrightarrow S_{\text{cl}}^{m+\frac{n}{4}-\frac{k}{2}}(N^*M; \Omega_{\text{fibre}}) / S^{m+\frac{n}{4}-\frac{k}{2}-1}. \tag{9.67}$$

Here the subscript cl indicates the requirement that the supports of these section of the fibre-density bundle on N^*M should project to a subset of M which is the intersection with X of a closed subset. If M is itself a closed submanifold this condition is void.

LEMMA 9.36. *For any embedded submanifold of codimension k in a manifold of dimension n and any $m \in \mathbb{R}$*

$$0 \hookrightarrow I^{m-1}(X, M) \hookrightarrow I^m(X, M) \overset{\sigma_m}{\to} \longrightarrow S_{\mathrm{cl}}^{m+\frac{n}{4}-\frac{k}{2}}(N^*M; \Omega_{\mathrm{fibre}})/S_{\mathrm{cl}}^{m+\frac{n}{4}-\frac{k}{2}-1} \longrightarrow 0 \tag{9.68}$$

is exact.

PROOF. The surjectivity of the symbol map follows by the use of a partition of unity. If $a \in S_{\mathrm{cl}}^{m+\frac{n}{4}-\frac{k}{2}}(N^*M; \Omega_{\mathrm{fibre}})$ then, by definition, its support is a subset of M which is of the form $M \cap F$ for some closed subset $F \subset X$. Take a covering of $\operatorname{supp} a$ by coordinate charts in which M takes the local form (9.2) and a partition of unity, ϕ_j^2, subordinate to the covering of X by these coordinate charts and $X \setminus \operatorname{supp}(a)$. Then for each j we can use the local representation, obtained by inverse Fourier transform from (9.33), to find an element $u_j \in I^m(X, M)$ with symbol $\phi_j a$. Taking

$$u = \sum_j \phi_j u_j \tag{9.69}$$

gives an element of $I^m(X, M)$ with symbol a. The proof of the remainder of the exactness of (9.68) is similar. □

The results on the pull-back and push-forward of conormal distributions in Chapter 5 also have analogues for the conormal distributions associated to a submanifold. We shall defer the discussion of these operations until after the introduction of Lagrangian distributions and then give results in that wider context. For the moment, with a view to that later generalization, we simply note how the conormal distributions associated to a submanifold are themselves obtained by push-forward of conormal distributions associated to a hypersurface. Indeed this is implicit in the local representation (9.33):

$$u(y, z) = (2\pi)^{-k} \int_{\mathbb{R}^k} e^{iy\cdot\eta} a(z, \eta) d\eta. \tag{9.70}$$

We can freely assume that a is supported in $|\eta| > 1$, affecting u only by a smooth term. Then the introduction of polar coordinates in $\eta = r\tau$, $|\tau| = 1$ reduces (9.70) to

$$u(y, z) = (2\pi)^{-k} \int_0^\infty \int_{\mathbb{S}^{k-1}} e^{ir(y\cdot\tau)} a(z, r\tau) r^{k-1} d\tau dr. \tag{9.71}$$

As a function on $Y \times [0, \infty)$, where $Y = \mathbb{R}^{n-k} \times \mathbb{S}^{k-1}$, $a(z, r\tau)$ is a symbol. Thus (9.71) can be written as a push-forward:

$$\begin{gathered} u = \pi_*(v) \text{ where } \pi(y, z, \tau) = (y, z) \text{ and} \\ v = (2\pi)^{-k} \int_0^\infty e^{ir(y\cdot\tau)} a(z, r\tau) r^{k-1} dr. \end{gathered} \tag{9.72}$$

Now $f = y \cdot \tau \in \mathcal{C}^\infty(\mathbb{R}^k \times \mathbb{S}^{k-1})$ vanishes precisely on the submanifold $Z = \{(y, z, \tau); y \perp \tau\}$ where its differential is non-vanishing. In fact it follows directly from (9.72) that $v \in I^m(\mathbb{R}^n \times \mathbb{S}^{k-1}, Z)$.

Remarks: Add topology (say for H^s-based spaces), density of $\mathcal{C}^\infty$ and pull-back under maps transversal to the submanifold.

CHAPTER 10

The scattering amplitude

We have already defined, in (8.72), the scattering kernel $\kappa_V \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ corresponding to a smooth potential with compact support, $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, for $n \geq 3$ odd. We now proceed to analyse its basic properties, support, singular support and asymptotic behaviour. To investigate the latter we shall use the Lax-Phillips semigroup.

Recall that we have shown the initial value problem

$$\begin{gathered}(D_t + D_s)k(t,s,\omega) + V_{\mathrm{LP}}\cdot k(t,s,\omega) = 0\\ k(0,s,\omega) = k_0(s,\omega)\end{gathered} \tag{10.1}$$

to have a unique solution for each $k_0 \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. Indeed the solution is just $k(t,x,\omega) = W(t)k_0(s,\omega)$ which in turn can be written

$$k(t,s,\omega) = \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} E_{\mathrm{LP}}(t,s,\omega;s',\theta)k_0(s',\theta)ds'd\theta. \tag{10.2}$$

The solution is $\mathcal{C}^\infty$ and the support properties of E_{LP} found in Lemma 8.29 show that

$$\begin{gathered}k_0(s,\omega) = 0 \text{ in } s \leq -\rho \Longrightarrow k(t,s,\omega) = 0 \text{ in } s \leq -\rho\ \forall\ t \geq 0\\ k_0(s,\omega) = 0 \text{ in } s \leq \rho \Longrightarrow k(t,s,\omega) = 0 \text{ in } s \leq \rho\ \forall\ t \geq 0.\end{gathered} \tag{10.3}$$

Observe that the quotient of these two spaces of initial data

$$\mathcal{K}^\infty = \frac{\left\{k_0 \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}); k_0(s,\omega) = 0 \text{ in } s \leq -\rho\right\}}{\left\{k_0 \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1}); k_0(s,\omega) = 0 \text{ in } s \leq \rho\right\}} \tag{10.4}$$

can be identified with the space

$$\mathcal{K}^\infty \equiv \left\{h \in \mathcal{C}^\infty((-\infty,\rho]\times\mathbb{S}^{n-1}); h(s,\omega) = 0 \text{ in } s \leq -\rho\right\} \tag{10.5}$$

by the obvious restriction map, $h = k_0|_{\{s\leq\rho\}}$.

Lemma 10.37. *For each* $t \geq 0$

$$Z(t)h = W(t)k_0|_{\{s\leq\rho\}},\ \textit{where}\ h = k_0|_{\{s\leq\rho\}}, \tag{10.6}$$

defines a semigroup (the Lax-Phillips semigroup) of operators on $\mathcal{K}^\infty$*; this semigroup extends by continuity to a bounded semigroup of operators on the Hilbert space*

$$\mathcal{K} = \left\{h \in L^2((-\infty,\rho]\times\mathbb{S}^{n-1}); h(s,\omega) = 0 \textit{ in } s < -\rho\right\}. \tag{10.7}$$

Proof. That $Z(t)$ is well defined follows from (10.4) since if k_0' is another representative of h then $k_0' - k_0$ vanishes in $s \leq \rho$ and hence so does $W(t)(k_0' - k_0)$, for any $t \geq 0$. A similar argument shows that $Z(t)$ is a semigroup too, since if k_0

is a representative of h then $W(t)k_0$ is a representative of $Z(t)h$, so $Z(s)Z(t)h = W(s)[W(t)k_0]|_{s\le\rho} = Z(t+s)h$.

The boundedness of $W(t)$ on L^2 implies the boundedness of $Z(t)$ on $\mathcal{K}$. □

Notice that

$$\mathcal{K} \equiv L^2((-\rho,\rho)\times\mathbb{S}^{n-1}). \tag{10.8}$$

We did not emphasize this initially since the action of $Z(t)$ on $\mathcal{K}$ includes a 'boundary condition' of vanishing near $s=-\rho$ which is well described by the definition (10.5) and the support properties of the kernel of $W(t)$.

One reason that the semigroup $Z(t)$ is so useful is that, for scattering by a compactly-supported potential, it eventually has a smooth kernel.

LEMMA 10.38. *For any $t > 2\rho$ the operator $Z(t)$ has a $\mathcal{C}^\infty$ kernel which depends smoothly on $t \in (2\rho,\infty)$.*

PROOF. The action of $Z(t)$ can always be written in terms of the kernel of $W(t)$, i.e.

$$Z(t)h(s,\omega) = \int_{[-r,\rho]\times\mathbb{S}^{n-1}} E_{\mathrm{LP}}(t,s,\omega;s',\theta)h(s',\theta)ds'd\theta. \tag{10.9}$$

Now, by Lemma 8.28, $E_{\mathrm{LP}}(t,s,\omega,s',\theta)$ is $\mathcal{C}^\infty$ in $t>2\rho$, $|s|,|s'|\le\rho$. □

The smoothness of the Schwartz kernel of $Z(t)$, for $t>2\rho$ means that $Z(t)$ is then compact as an operator on $\mathcal{K}$. Recall that a compact operator on a Hilbert space is a bounded operator, B, which is norm-approximable by operators of finite rank, i.e. there exists a sequence B_n satisfying

$$\dim[B_n(\mathcal{K})] < \infty \ \forall\ n,\ \|B - B_n\| \longrightarrow 0 \text{ as } n\to\infty. \tag{10.10}$$

Here the operator norm is just

$$\|B\| = \sup_{\|f\|_{\mathcal{K}}=1} \|Bf\|_{\mathcal{K}}. \tag{10.11}$$

For any operator with smooth kernel

$$\|B\| \le C \sup_{[-\rho,\rho]\times\mathbb{S}^{n-1}\times[-\rho,\rho]\times\mathbb{S}^{n-1}} |B(s,\omega,s',\theta)|. \tag{10.12}$$

Thus the compactness, on $\mathcal{K}$, of any operator with smooth kernel follows from the approximability of such smooth functions by polynomials on $\mathbb{R}^{2n+2}$ in the uniform norm on $[-\rho,\rho]\times\mathbb{S}^{n-1}\times[-\rho,\rho]\times\mathbb{S}^{n-1}$, i.e. the Stone-Weierstrass theorem.

The compactness of $Z(t)$ for $r > 2\rho$ is important because it implies that the semigroup has an asymptotic expansion in exponentials. To understand where this comes from consider the special case of a semigroup on a space of finite dimension. This is always of the form e^{iAt} for some matrix A. The kernel of the semigroup (which in the finite rank case always extends to a group) is then of the form

$$Z(t) = \sum_j \sum_{i\le N_j} e^{i\lambda_j t}\left(\sum_{k\le i}\frac{t^k}{k!}e_{kj}\right)\otimes f_{ij} \tag{10.13}$$

where the e_{ij} form a basis of generalized eigenfunctions for A corresponding to the eigenvalue λ_j and the f_{ij} are an eigenbasis for A^*.

PROPOSITION 10.24. *Let $Z(t)$ be a bounded semigroup of operators on a Hilbert space $\mathcal{K}$ and suppose that, for some $T > 0$, $Z(T)$ is compact, then there are sequences $\lambda_j \in \mathbb{C}$, with $\Im\lambda_j \to \infty$ as $j \to \infty$, $N_j \in \mathbb{N}$ and elements e_{ij}, $f_{ij} \in \mathcal{K}$ for $i \leq N_j$, $j \in \mathbb{N}$ such that for any J the difference*

$$Z_J(t) = Z(t) - \sum_{j\leq J}\sum_{i\leq_j} e^{i\lambda_j t}\left(\sum_{k\leq i}\frac{t^k}{k!}e_{kj}\right)\otimes f_{ij} \tag{10.14}$$

is a bounded semigroup of operators on $\mathcal{K}_J$, the space annihilated by all the F_{ij} with $j \leq J$ and with

$$\sup_{t\geq 0} e^{(\Im\lambda_J-\epsilon)t}\|Z_J(t)\| < \infty \ \forall\ \epsilon > 0. \tag{10.15}$$

PROOF. This result is a simple consequence of the spectral properties of compact operators. Recall that the spectral radius of a bounded operator is just

$$\operatorname{Rad}(B) = \limsup \|B^n\|^{1/n}. \tag{10.16}$$

It is defined so that the Neumann series for the resolvent operator

$$(z - B)^{-1} = z^{-1}\sum_{j\leq 0} z^{-j}B^j \tag{10.17}$$

converges precisely in the region $|z| > \operatorname{Rad}(B)$. In fact the converse is also true, namely $\operatorname{Rad}(B)$ is the radius of the smallest disk centered at the origin with complement on which the resolvent extends to be holomorphic.

If B is compact then for any $\epsilon > 0$ it has a decomposition

$$B = B_\epsilon + Q_\epsilon,\ \operatorname{Rad}(Q_\epsilon) \leq \epsilon, \tag{10.18}$$

(10.19)

where B_ϵ is of finite rank and commutes with any operator commuting with B. Of course we always have $\operatorname{Rad}(B) \leq \|B\|$ so the important point here is the commutativity since otherwise (10.19) follows from the definition of compactness in (10.10).

In the interests of completeness we pause to prove this result. Let us start with the earlier statement about the spectral radius. We know that $(z - B)^{-1}$ is holomorphic in $|z| > \operatorname{Rad}(B)$ for any bounded operator B. Suppose then that it is holomorphic, with values in the bounded operators, in $|z| > r$, for some $r < \operatorname{Rad}(B)$. Using Cauchy's theorem we find that

$$\operatorname{Id} = \frac{1}{2\pi i}\int_{|z|=r'}(z - B)^{-1}dz, \quad \forall\ r' > r. \tag{10.20}$$

Indeed, Cauchy's theorem implies that the integral is independent of $r' > r$. From the uniform convergence of (10.17) near infinity, only the first term contributes as $r' \to \infty$ so (10.20) follows. From a similar argument, or repeated use of the identity $(z - B)(z - B)^{-1} = \operatorname{Id}$, it follows that

$$B^k = \frac{1}{2\pi i}\int_{|z|=r'} z^k(z - B)^{-1}dz, \quad \forall\ r' > r,\ k \in \mathbb{N}. \tag{10.21}$$

Directly estimating the integral shows that $\operatorname{Rad}(B) \leq r$.

Having identified the spectral radius with the radius of the smallest disk around the origin such that $(z-B)^{-1}$ extend holomorphically to the complement, we can prove (10.19). Starting from (10.10) choose an operator of finite rank F such that $B-F$ has norm at most ϵ. Let $\mathcal{K}_1 \subset \mathcal{K}$ be a subspace of finite dimension containing the range of F and such that $\mathcal{K}_2 = \mathcal{K}_1^\perp$ is contained in the null space of F. Then consider the decomposition of B using the orthogonal projections π_i onto the $\mathcal{K}_i$:

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2,\ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},\ B_{ij} = \pi_i B \pi_j. \tag{10.22}$$

Here all but B_{22} are certainly finite rank and $\|B_{22}\| \le \epsilon$.

In terms of the decomposition of $(z-B)^{-1} = I(z)$ corresponding to (10.22) we need

$$\begin{aligned} &(z-B_{11})I_{11} + B_{12}I_{21} = \pi_1,\ (z-B_{11})I_{12} + B_{12}I_{22} = 0 \\ &B_{21}I_{11} + (z-B_{22})I_{21} = 0,\ B_{21}I_{12} + (z-B_{22})I_{22} = \pi_2. \end{aligned} \tag{10.23}$$

Now the resolvent $(z-B_{22})^{-1}$ is holomorphic in $|z| > \epsilon$, so the last equation can be solved there for I_{22} in terms of I_{12}. Substituting into the second equation gives

$$\left[z - B_{11} - B_{12}(z-B_{22})^{-1}B_{21}\right] I_{12} = -B_{12}(z-B_{22})^{-1}\pi_2. \tag{10.24}$$

Both sides here map $\mathcal{K}_2$ to $\mathcal{K}_1$. The first factor on the left is a bounded map on $\mathcal{K}_1$. It is invertible for $|z|$ large and is holomorphic in $|z| > \epsilon$. It follows that the inverse, and hence also I_{12}, is meromorphic in $|z| > \epsilon$. From this, and a similar treatment of the first two equations, it follows that $I(z)$, the resolvent of B, is meromorphic in $|z| > \epsilon$. Such singularities as it has in the region $\epsilon < |z| \le \mathrm{Rad}(B)$ must be poles of finite order (i.e. if there is a pole at $z = z'$ then $(z-z')^N I(z)$ is holomorphic near z' for some $N \in \mathbb{N}$) and of finite rank. This means that in the Laurent series expansion at any pole

$$I(z) = \sum_{j \ge -N} (z-z')^j Q_j \text{ the } Q_j \text{ for } j < 0 \text{ have finite rank.} \tag{10.25}$$

Thus we have shown that for any compact operator the resolvent $I(z)$ extends to be meromorphic in $\mathbb{C} \setminus 0$ with poles of finite rank. The decomposition (10.19) now follows from (10.21) for $k = 1$ by writing

$$Q_\epsilon = \frac{1}{2\pi i} \int_{|z|=\epsilon'} z(z-B)^{-1} dz \tag{10.26}$$

where $\epsilon' < \epsilon$ is chosen so that there are no poles on the circle $|z| = \epsilon'$. The difference

$$B_\epsilon = B - Q_\epsilon = \frac{1}{2\pi i}\left[\int_{|z|=\mathrm{Rad}(B)+1} - \int_{|z|=\epsilon'} \right] z(z-B)^{-1} dz \tag{10.27}$$

can be evaluated by residues, showing that it is of finite rank. Since $I(z)$ commutes with any operator commuting with B the same follows for B_ϵ. Moreover, directly from (10.26) it is clear that $\mathrm{Rad}(Q_\epsilon) \le \epsilon$. Thus we have proved (10.19).

In fact this approach gives rather more. The operator

$$\pi_\epsilon = \frac{1}{2\pi i}\left[\int_{|z|=\mathrm{Rad}(B)+1} - \int_{|z|=\epsilon'} \right] (z-B)^{-1} dz \tag{10.28}$$

is a projection. This follows by use of the resolvent identity:

$$(z-B)^{-1}\circ(z'-B)^{-1}=(z-z')^{-1}[(z-B)^{-1}-(z'-B)^{-1}] \tag{10.29}$$

valid whenever $z\neq z'$ are not poles of the resolvent. If (10.28) is split into a finite sum of terms each of which is the same integral along a small contour enclosing just one of the poles of the resolvent in the annulus $\epsilon<|z|\le \operatorname{Rad}(B)$ then these terms commute and each is a projection. Since these projections commute with $Z(t)$ for all $t\ge 0$

$$Z_\epsilon(t)=\pi_\epsilon Z(t)\pi_\epsilon \tag{10.30}$$

is a semigroup for each $\epsilon>0$. This is a semigroup of finite rank, so has an expansion as in (10.13). Thus the sum on the right in (10.14) just arises from the kernels of these finite rank semigroups.

To complete the proof of Proposition 10.24 it is only necessary to note that the terms in the expansion of Z_ϵ so obtained must be independent of ϵ. □

Using the Lax-Phillips semigroup and the analysis already made of the singularities we can now find the basic properties of the scattering kernel.

Proposition 10.25. *If $n\ge 3$ is odd then the scattering kernel κ_V corresponding to $V\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies*

$$\operatorname{supp}(\kappa_V)\subset[-2\rho,\infty)\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1} \text{ if } \operatorname{supp}(V)\subset\{|x|\le\rho\}. \tag{10.31}$$

$$\kappa_V\in I^{\frac14}(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1},D),\ D=\{t=0,\theta=\omega\} \tag{10.32}$$

and κ_V has a complete asymptotic expansion as $t\longrightarrow\infty$

$$\kappa_V(t,\omega,\theta)\sim\sum l\le N(j) \tag{10.33}$$

$$j=1,\dots,\infty a_{j,l}(\omega,\theta)e^{i\lambda_j t}t^l \quad \text{as } t\longrightarrow\infty \tag{10.34}$$

with $a_{j,l}\in\mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and $\Im\lambda_j\longrightarrow\infty$ as $j\longrightarrow\infty$.

Here the asymptotic expansion is in $\mathcal{C}^\infty$, i.e. if the λ_j are ordered so that $\Im\lambda_j$ is a non-decreasing sequence then (10.34) means that for any $p,J\in\mathbb{N},\epsilon>0$ and any $\mathcal{C}^\infty$ differential operator P on $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ there exists a constant C such that

$$\Big|D_t^pP\Big[\kappa_V(t,\omega,\theta)-\sum l\le N(j) \tag{10.35}$$

$$j<Ja_{j,e}(\omega,\theta)e^{i\lambda jt}t^l\le C\exp((\Im\lambda_J-\epsilon)t)\quad t>1. \tag{10.36}$$

Note also that (10.32) implies in particular that

$$\operatorname{singsupp}(K_V)\subset\{t=0,\theta=\omega\}. \tag{10.37}$$

These three results give rather precise information on the growth and regularity of κ_V. Before proceeding to the proof we note the following consequence.

Corollary 10.3. *The scattering amplitude*

$$a_V(\omega,\theta,\lambda)=\int\exp(-i\lambda t)\left[\kappa_V(t,\omega,\theta)-\delta(t)\delta_\theta(\omega)\right]dt \tag{10.38}$$

is a meromorphic function of $\lambda\in\mathbb{C}$, holomorphic in some half-plane $\Im\lambda<C'$, with values in $\mathcal{C}^{-\infty}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$.

from -250 to 250, y from -200 to 250 .5pt ¡4pt¿ [.2,.67] from -200 0 to 200 0 ¡4pt¿ [.2,.67] from 0 -180 to 0 180 -150 -150 150 150 / from

FIGURE 1. Support of α'

We shall actually show below that a_V is meromorphic, with poles at the λ_j in (10.34), with values in $\mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ and then analyse its behaviour as $|\Re\lambda| \longrightarrow \infty$ with $|\Im\lambda|$ bounded.

PROOF. Proof of Proposition 10.25 We can represent κ_V, restricted to any finite strip $t < T$, in terms of a solution to the transformed wave equation and hence, ultimately, in terms of a solution to the original wave equation. By definition κ_V is obtained from the solution of the continuation problem

$$\begin{gathered}(D_t + D_s - V_{\mathrm{LP}})\alpha = 0 \text{ in } \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1} \\ \alpha = \delta(s - t)\delta_\theta(\omega) \text{ in } t << 0\end{gathered} \tag{10.39}$$

in the form

$$\alpha(t, s, \omega; \theta) = \kappa_V(t - s, \omega; \theta) \text{ in } s \geq \rho. \tag{10.40}$$

The solution to (10.39) can be reduced to a forcing problem for the original wave equation. Namely if $w \in \mathcal{C}^{-\infty}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1})$ satisfies

$$\begin{gathered}(D_t^2 - \Delta - V)w = c_n V \cdot D_t^{\frac{n-3}{2}} \delta(t - x \cdot \theta), \quad c_n = 2^{-\frac{1}{2}}(2\pi)^{-\frac{n}{2}} \\ w = 0 \text{ in } t << 0\end{gathered} \tag{10.41}$$

then

$$\alpha(t, s, \omega, \theta) = \delta(t - s)\delta_\theta(\omega) + LP\begin{pmatrix} w(t) \\ D_t w(t)\end{pmatrix}. \tag{10.42}$$

Indeed, the results of Chapter 8 show that if w satisfies (10.41) then it has compact x-support and

$$\alpha' = LP\begin{pmatrix} w(t) \\ D_t w(t)\end{pmatrix} \in \mathcal{C}^{-\infty}(\mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \tag{10.43}$$

satisfies

$$(D_t + D_s - V_{\mathrm{LP}})\alpha' = R_n \cdot V \cdot D_t^{\frac{n-3}{2}} \delta(t - x \cdot \theta) = V_{\mathrm{LP}}\delta(s - t)\delta_\theta(\omega). \tag{10.44}$$

Thus α defined by (10.42), which is just $\alpha' + \delta(s - t)\delta_\theta(\omega)$, satisfies (10.39). The formulae (10.40), (10.42) therefore hold by the uniqueness of the solution to (10.39).

The support condition, (10.31), is easily deduced from (10.44). Since V_{LP} has support in $|s|$, $|s'| \leq \rho$, and $\delta(s - t)$ has support on $s = t$ from (10.44)

$$(D_t + D_s - V_{\mathrm{LP}})\alpha' = 0 \text{ in } t < -\rho \text{ and } s > \rho. \tag{10.45}$$

Since $\alpha' = 0$ in $t << 0$ uniqueness for the Cauchy problem shows that $\alpha' = 0$ in $t < -\rho$; it must therefore also vanish in $\{s > \rho, t < s - 2\rho\}$, see Figure 1. From (10.40) it follows that κ_V vanishes in $t < -2\rho$.

Next we locate the singularities of α', just as we found the singular support of the forward fundamental solution of $D_t + D_s - V_{\mathrm{LP}}$. Notice that if u satisfies the continuation problem

$$\begin{gathered}(D_t^2 - \Delta - V)u = 0 \\ u = \delta(t - x \cdot \theta) \quad t << 0\end{gathered} \tag{10.46}$$

then, from the uniqueness of the solution to (10.41)

$$w = c_n D_t^{\frac{n-3}{2}} [u - \delta(t - x \cdot \theta)]. \tag{10.47}$$

This reduces the question to a continuation problem of the type we considered in Chapter 1. Using the results obtained there, in particular Proposition 1.1, together with the removal of the $\mathcal{C}^\infty$ error term in Chapter 6, we know that

$$u = \delta(t - x \cdot \theta) + H(t - x \cdot \theta) g(t, x, \theta) + \phi(t, x, \theta) \tag{10.48}$$

where $g, \phi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1})$ have x-compact supports.

Together, (10.40), (10.42) and (10.47) show how to calculate κ_V from u. Observe that the $\mathcal{C}^\infty$ term ϕ contributes a $\mathcal{C}^\infty$ term to w in (10.47), hence a $\mathcal{C}^\infty$ term to α in (10.42) and so finally a $\mathcal{C}^\infty$ term to κ_V. Thus in computing the singularities of κ_V it can be ignored. Setting $u' = H(t - x \cdot \theta) g(t, x, \theta)$ we see that

$$\alpha'(t, s, \theta, \omega) \equiv c_n^2 \left\{ D_s^{\frac{n-1}{2}} R D_t^{\frac{n-1}{2}} u' - D_s^{\frac{n+1}{2}} R D_t^{\frac{n-3}{2}} u' \right\} \quad \text{mod } \mathcal{C}^\infty. \tag{10.49}$$

It will turn out that those two terms are the same, so let us concentrate on the first, which can be written

$$c_n^2 D_s^{\frac{n-1}{2}} D_t^{\frac{n-1}{2}} R u'. \tag{10.50}$$

Clearly then we need to examine the Radon transform of u'. This we write in the form

$$R u' = \pi_* \beta^* u' \tag{10.51}$$

where

$$\begin{array}{ccc} & M & \pi\beta \\ \mathbb{R}_t \times \mathbb{R}_s \times \mathbb{S}^{n-1}_\omega \times \mathbb{S}^{n-1}_\theta & & \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{S}^{n-1}_\theta \end{array} \tag{10.52}$$

are the two projections from $M = \{(t, s, x, \omega, \theta); s = x \cdot \omega\}$.

Now u' is conormal with respect to $\{t = x \cdot \theta\} = H$. We have already checked the transversality of β to H. In any case this is just the obvious statement

$$d\beta^*(t - x \cdot \theta) \neq 0 \text{ on } M \cap \beta^{-1}(H). \tag{10.53}$$

As before, the fibres of π are the $(n-1)$ dimensional planes in M given by $x \cdot \omega = s$, with ω, s, θ and t fixed. Thus the critical set of $\beta^{-1}(H)$ with respect to π is

$$C_H = \{(t, s, x, \omega, \theta) \in \beta^{-1}(H); d_x(t - x \cdot \theta) = c d_x(s - x \cdot \omega)\}. \tag{10.54}$$

Thus again applying Proposition 5.13

$$\text{singsupp}(Ru') \subset \pi(C_H) = \{s = \sigma t, \theta = \sigma \omega, \sigma = \pm 1\}. \tag{10.55}$$

The same argument applies to the second term in (10.49) so

$$\text{singsupp}(\alpha') \subset \{t = s, \theta = \omega\} \cup \{t = -s, \theta = -\omega\}. \tag{10.56}$$

In $s > \rho$ $(D_t + D_s)\alpha' = 0$. There can therefore be no singularities of the second type in this region, since they would propagate along the lines $t + s =$const. Thus

$$\text{singsupp}(\alpha') \subset \{t = s, \theta, \omega, s \geq -\rho\} \cup \{t = -s, \theta = -\omega, |s| \leq \rho\}. \tag{10.57}$$

Since κ_V is given by (10.40) this proves (10.37).

To strengthen this to the statement, (10.32), of conormal regularity simply write (10.51) explicitly in the form

$$\begin{aligned} Ru' &= R\int_{\mathbb{R}} e^{i(t-x\cdot\theta)\tau} b(t,x,\theta,\tau)d\tau \\ &= \int_{\omega^\perp}\int_{\mathbb{R}} e^{i(t-(s\omega+x')\cdot\theta)\tau} b'(t,s\omega+x',\theta,\tau)d\tau dx' \end{aligned} \tag{10.58}$$

where b is a symbol of order . In terms of the fibre variables $\zeta = (\tau\omega + x'\tau) \in \mathbb{R}^n$ the phase function can be written

$$(t - s\omega\cdot\theta - x'\cdot\theta)\tau = (t\omega - s(\theta\cdot\omega)\omega + \theta)\cdot\zeta. \tag{10.59}$$

Consider the map

$$\begin{aligned} G &: \mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1} \\ G(t,s,\omega,\theta) &\longmapsto (t\omega - s(\theta\cdot\omega)\omega+\theta, t, s, \omega, \theta) = (Z,t,s,\omega,\theta). \end{aligned} \tag{10.60}$$

This is clearly transversal to $Z = 0$, which pulls back to the submanifolds $\{t = 0, \theta = \pm\omega\}$. Moreover $b(t,s,\omega,\theta,\zeta) = b'(t, s\omega + x', \theta, \tau)$ is a symbol in ζ so

$$u''(Z,s,\omega,\theta) = \int_{\mathbb{R}^n} e^{iZ\cdot\zeta} b(t,s,\omega,\theta,\zeta)d\zeta \tag{10.61}$$

is conormal with respect to $Z = 0$. Thus $Ru' = G^*u''$ is conormal as stated in (10.32).

Remark: This needs to be tidied up a bit and related to the results, still to be added, on smooth approximability and pull-back from Chapter 9.

Finally then we need to prove (10.34). In (10.44) the right-hand side vanishes for $t > \rho$ so α' satisfies the homogeneous equation. Since $\alpha' = 0$ in $s < -\rho$ its restriction to $s < \rho$ is given by the Lax-Phillips semigroup:

$$\alpha'(t,s,\omega,\theta) = Z(t-\rho)\alpha'(\rho,s,\omega,\theta) \text{ in } s < \rho,\ t > \rho. \tag{10.62}$$

Moreover from (10.56) $\alpha'(\rho,s,\omega,\theta)$ is itself $\mathcal{C}^\infty$. Thus α' has an asymptotic expansion coming from (10.14) as $t \to \infty$. From this the expansion for κ_V follows using (10.40). This completes the proof of Proposition10.25. □

Using these results, and the formula (10.49) we can obtain more detailed information about the scattering amplitude than given in Corollary 10.3.

Proposition 10.26. *The scattering amplitude, defined by* (10.38) *for* $\Im\lambda << 0$, *extends uniquely to a* $\mathcal{C}^\infty$ *function*

$$a_V \in \mathcal{C}^\infty(\mathbb{S}^{n-1}_\omega \times \mathbb{S}^{n-1}_\theta \times [\mathbb{C}\backslash\{\lambda_j\}]_\lambda) \tag{10.63}$$

which is meromorphic in λ *with poles of finite rank (and order) and satisfies, for some* C *and any* $\mathcal{C}^\infty$ *differential operator* P *on* $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$

$$\sup_{\Im\lambda\le C} \left|\exp(2\rho\Im\lambda)(1+|\Re\lambda|)^{-n+2} P a_V\right| < \infty. \tag{10.64}$$

Proof. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ be identically equal to 1 in a neighbourhood of $t = 0, \theta = \omega$, but vanish in $|t| > \frac{\rho}{10}$, $|\theta - \omega| > \frac{1}{10}$. Then the decomposition

$$\kappa_V = \kappa_V' + \kappa_V'',\ \kappa_V' = \phi\kappa_V \tag{10.65}$$

has second term, κ''_V, $\mathcal{C}^\infty$, with κ'_V having compact support in $|t| \le \frac{\rho}{10}$. Thus κ''_V still satisfies (10.31) and (10.34). The Paley-Wiener theorem shows that its Fourier transform

$$a''_V(\omega,\theta,\lambda) = \int e^{-i\lambda t}(1-\phi(t,\omega,\theta))\kappa_V(t,\omega,\theta)dt \tag{10.66}$$

satisfies (10.63) and an even stronger form of (10.64), namely

$$\sup_{\Im\lambda < C} \tag{10.67}$$

$$\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\left|\exp(2\rho\Im\lambda)(1+|\Re\lambda|)^k P a''_V\right| < \infty \quad \forall\ P, k. \tag{10.68}$$

Now, from the compactness of its support, the Fourier transform of κ'_V, a'_V, is entire in $\lambda\in\mathbb{C}$. We shall show that, for some $R < 2\rho$,

$$\begin{cases} a'_V = \int e^{i\lambda t}\phi\kappa_V(t,\omega,\theta)dt \in \mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{C}) \\ \left|\exp(R\Im\lambda)(1+|\lambda|)^{-n+2}Pa'_V\right| < \infty \text{ in } \Im\lambda < C \end{cases} \tag{10.69}$$

This will certainly imply (10.64) for $a_V = a'_V + a''_V$. Again, by the Paley-Wiener Theorem, adding a $\mathcal{C}^\infty$ term to κ_V, and hence to κ'_V, contributes an entire term to a'_V satisfying the estimates in (10.69) and rapidly decreasing as $\Re|\lambda| \longrightarrow \infty$. Thus we can replace κ'_V by ϕ times the right side of (10.49), evaluated at $s=\rho$, setting

$$\begin{aligned} \tilde{a}_V(\omega,\theta,\lambda) &= \int e^{-i\lambda t}\tilde{\kappa}_V(t,\omega,\theta)dt \\ \tilde{\kappa}_V(t-\rho,\omega,\theta) =& C_2\big\{D_s^{\frac{n-1}{2}}D_t^{\frac{n-1}{2}}\phi Ru' \\ &- D_s^{\frac{n-1}{2}}D_t^{\frac{n-3}{2}}fRu'\big\}\big|_{s=\rho}(t-\rho,\omega,\theta). \end{aligned} \tag{10.70}$$

We are therefore reduced to computing Ru' in more detail! Set

$$\omega^\perp = \{\xi\in\mathbb{R}^n; \xi\cdot\omega = 0\}, \tag{10.71}$$

a subspace of dimension $n-1$. Then, since $|\theta-\omega| < \frac{1}{10}$ on the support of f, we can set

$$\theta = d(\xi)\omega + \xi \quad |\xi| < \frac{1}{2}, \xi\in\omega^\perp, d(\xi) = (1-|\xi|^2)^{\frac{1}{2}}. \tag{10.72}$$

Similarly

$$x\cdot\omega = s \Longrightarrow x = s\omega+\eta, \quad \eta\in\omega^\perp. \tag{10.73}$$

If $d\eta$ denotes the Euclidean measure on $\omega^\perp$ then we can write

$$\phi Ru' = \phi\int H(t-d(\xi)s-\eta\cdot\xi)\cdot g(t,s\omega+\eta,d(\xi)\omega+\xi)d\eta. \tag{10.74}$$

Now let b be the Fourier transform

$$b(x,\omega,\theta,\tau) = \int e^{-i\tau(t-x\cdot\theta)}H(t-x\cdot\theta)\phi(t,\omega,\theta)g(t,x,\theta)dt. \tag{10.75}$$

We know that b is a symbol in τ of order -1. Inserting this in (10.74) we have

$$\begin{aligned} \phi Ru' &= \frac{1}{2\pi}\int e^{i\tau(t-x\cdot\theta)}b(x,\omega,\theta,\tau)d\tau d\eta \\ &= \frac{1}{2\pi}\int e^{i\tau(t-d(\xi)s-\eta\cdot\xi)}b(s\omega+\eta,\omega,d(\xi)\omega+\xi,\tau)d\tau d\eta. \end{aligned} \tag{10.76}$$

To compute the form of $\tilde{a}_V$ from this we need to use (10.70). Carrying out the differentiation replaces b by another symbol, of order $n-2$ $(=-1+n-1)$, so

$$\tilde{\kappa}_V(t-s,\omega,\theta)=\frac{c_n^2}{2\pi}\int e^{i\tau(t-d(\xi)s-\eta\cdot\xi)}\tilde{b}(s,\omega,\xi,\eta,\tau)d\tau d\eta \quad (s>\rho)$$

$$\tilde{b}(s,\omega,\xi,\eta,\tau)=\tau^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{j}(-\tau d(\xi))^{\frac{n-1}{2}-j}D_s^j b \qquad (10.77)$$

$$-\tau^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-3}{2}}\binom{\frac{n-3}{2}}{j}(-\tau d(\xi))^{\frac{n-3}{2}-j}D_s^j b.$$

Introducing $t+s$ in place of t in (10.77) gives

$$\tilde{a}_V(\omega,\theta,\lambda)=c_n^2\int e^{i[\lambda(1-d(\xi))s-\eta\cdot\xi]}\tilde{b}(s,\omega,\xi,\eta,\lambda)d\eta \quad (s\geq\rho). \qquad (10.78)$$

This proves the smoothness of $\tilde{a}_V$ and the estimate (10.69), since $\tilde{b}$ has compact support in $|\eta|\leq 2\rho$ (see (10.75)), is a symbol of order $n-2$, and $|(1-d(\xi))s-\eta-\xi|\leq\frac{\rho}{2}$ on the support of $\tilde{b}$. This completes the proof of Proposition 10.26. □

Of course we should not stop here! The formula for $\tilde{a}_V$ gives a precise description of the asymptotic behaviour of a_V as $\Re\lambda\to\infty$, with $|\Im\lambda|$ bounded.

Proposition 10.27. *If $|\Im\lambda|<C$ then as $|\Re\lambda|\longrightarrow\infty$ $a_V(\omega,\theta,\lambda)$ is rapidly decreasing if $|\lambda||\theta-\omega|\longrightarrow\infty$. If $\theta-\omega=\frac{1}{\tau}\zeta-\frac{1}{\tau^2}\omega,\zeta\in\omega^\perp$, then as $\tau\longrightarrow\infty$, where $\lambda=\tau+i\Lambda$, a_V has a complete asymptotic expansion:*

$$a_V(\omega,(1-\frac{1}{\tau^2})\omega+\frac{1}{\tau}\zeta,\tau+i\wedge)\sim\sum_{j\leq n-2}\tau^j a^{\pm}_{v,j}(\omega,\zeta) \text{ as } \tau\to\pm\infty \qquad (10.79)$$

where

$$a^{\pm}_{v,n-2}(\omega,\zeta)=\pm\frac{1}{2(2\pi)^n}\int_{\omega^\perp}e^{-i\zeta\cdot\eta}(\int V(\eta+r\omega)dr)d\eta. \qquad (10.80)$$

Proof. The terms other than $\tilde{a}_V$ in a_V are already rapidly decreasing as $|\Re\lambda|\longrightarrow\infty$ with $|\Im\lambda|$ bounded, so it suffices to examine $\tilde{a}_V$ given by (10.78), which we write in the form

$$\tilde{a}_V(\omega,\theta,\lambda)=c_n^2\exp(i\lambda(1-d(\xi))s)\hat{\tilde{b}}(s,\omega,\xi,\zeta,\lambda) \qquad (10.81)$$

the hat denoting the Fourier transform in $\eta\in\omega^\perp$. The condition $|\lambda||\theta-\omega|\longrightarrow\infty$, i.e. $|\tau||\theta-\omega|\longrightarrow\infty$, just means $|\tau\zeta|\longrightarrow\infty$. Since $\tilde{b}$ has compact support in η its Fourier transform is indeed rapidly decreasing at real infinity. This gives the rapid decrease away from $\theta=\omega$.

Similarly, setting $\theta-\omega=\frac{1}{\tau}\zeta-\frac{1}{\tau^2}\omega$, i.e. $\xi=\frac{1}{\tau}\zeta$ with $\zeta\in\omega^\perp$, we get

$$\begin{aligned}&\tilde{a}_V(\omega,(1-\frac{1}{\tau^2})\omega+\frac{1}{\tau}\zeta,\tau+i\wedge)\\&=c_n^2\exp(i\frac{\tau+i\wedge}{\tau^2}s)\hat{\tilde{b}}(s,\omega,\frac{1}{\tau}\zeta,\frac{\tau+i\wedge}{\tau}\zeta,\tau+i\wedge).\end{aligned} \qquad (10.82)$$

The fact that $\hat{\tilde{b}}$ has an expansion in the last variable, together with Taylor series in the other variables, gives (10.79). The leading term is just

$$
\begin{aligned}
&a_{v,n-2}(\omega,\zeta) = c_n^2 \hat{\tilde{b}}^{\pm}_{n-2}(\rho,\omega,\theta,z),\\
&\text{if } \tilde{b} \sim \sum_{k\le n-2} \tilde{b}_k^{\pm}(s,\omega,\xi,\eta)\lambda^k, \lambda \longrightarrow \infty.
\end{aligned} \tag{10.83}
$$

Thus we need to complete the leading term of $\tilde{b}$, in (10.77). It is just

$$
\begin{aligned}
&\tilde{b}^{\pm}_{n-2}(s,\omega,\xi,\eta) = (-1)^{\frac{n-1}{2}} d(\xi)^{\frac{n-3}{2}} (d(\xi)+1) b^{\pm}_{-1}(s,\omega,\xi,\eta)\\
&\text{if } b(s,\omega,\xi,\eta,\lambda) \sim \sum_{j\le -1} b_j^{\pm}(s,\omega,\xi,\eta)\lambda^j \quad \text{as } \lambda \longrightarrow \infty.
\end{aligned} \tag{10.84}
$$

This reduces the computation to that of the leading term in b, from (10.75). In fact

$$
b^{\pm}_{-1}(x,\omega,\theta) = \pm\, g(t,x,\theta)|_{t=x\cdot\theta}\,. \tag{10.85}
$$

Referring back to the construction of g we conclude that

$$
b^{\pm}_{-1}(x,\omega,\theta) = \pm(-\frac{i}{2}) \int_{-\infty}^{\infty} V(x+(r-x\cdot\omega)\omega)dr \tag{10.86}
$$

Thus

$$
b^{\pm}_{-1}(\omega,0,\eta) = \mp\frac{i}{2} \int_{-\infty}^{\infty} V(\eta + r\omega)dr \tag{10.87}
$$

is just the average of V over ω-lines. This proves (10.80) and hence the proposition. □

Proposition 10.27 has one very important consequence. Namely it shows that the potential, V, can be recovered from the scattering amplitude a_V.

COROLLARY 10.4. *The scattering transform*

$$
\mathcal{C}^\infty_c(\mathbb{R}^n) \ni V \longmapsto a_V \in \mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times(\mathbb{C}\backslash\{\lambda_j\})) \tag{10.88}
$$

is, for $n \ge 3$ odd, injective.

PROOF. From Proposition 10.27 we can recover $a^{\pm}_{v,n-2}$ from a_V. Taking the inverse Fourier transform in ζ we conclude that

$$
\int_{-\infty}^{\infty} V(\eta + r\omega)dr \text{ is determined } \forall\ \omega\in\mathbb{S}^{n-1}, \eta\in\omega^{\perp}. \tag{10.89}
$$

This however is the *X-ray transform* of V, i.e. we have recovered the average of V along each line in $\mathbb{R}^n$. Since we can write

$$
\begin{aligned}
&\int_{x\cdot\theta=s} V dH_x = \int_Q (\int_{-\infty}^{\infty} V(r\omega+s\theta)dr)d\omega\\
&\text{where } Q=\{\omega\in\theta^{\perp};\omega\cdot\omega'=0\}
\end{aligned} \tag{10.90}
$$

for some fixed $\omega' \in \theta^{\perp}$ we can recover the Radon transform of V from a_V. The inversion formula for R completes the proof. □

The philosophical connection between the appearance here, in (10.89), of the x-ray transform of V and its appearance in X-ray tomography (CAT scans) deserves to be pondered a little. The occurence of the x-ray transform here can be traced to the fact that singularities propagate along straight lines, irrespective of the potential perturbation.

CHAPTER 11

Stationary scattering and spectral theory

Next we consider the relationship between the approach to scattering theory explored above, using progressing waves and the Radon transform, and other formulations. First the Fourier transform, in time, of the progressing wave solutions is analyzed, since this gives a complete set of generalized eigenfunctions for the continuous spectrum of $\Delta + V$. Similarly the Fourier transform of the forward fundamental solution of the wave equation gives the resolvent and so allows the 'limiting absorption principle,' to be proved. This analysis reproduces the results of classical 'stationary scattering theory' in this context and gives a detailed description of the spectrum of $\Delta + V$. The abstract scattering theory of Lax and Phillips is also described in this setting as is the radiation limit of Friedlander. These various approaches to scattering theory are also related to the existence and completeness of the Müller wave operators.

First we shall show that the the scattering amplitude as defined in Chapter 10 coincides with the scattering amplitude defined in the more classical, stationary, approach to scattering. In this approach the scattering amplitude is defined in terms of 'outgoing' generalized eigenfunctions of $\Delta + V$. These eigenfunctions are perturbations for large $|x|$ of the 'plane wave' generalized eigenfunctions of Δ of the form $\exp(i\lambda x \cdot \omega)$ where $\lambda \in \mathbb{R}$ is the frequency variable and $\omega \in \mathbb{S}^{n-1}$.

More precisely, for each real $\lambda \neq 0$ and each $\omega \in \mathbb{S}^{n-1}$, there is a unique function $\varphi(\lambda, x, \omega) \in \mathcal{C}^\infty(\mathbb{R}^n_x)$ such that

$$(\Delta + V - \lambda^2)\varphi = 0 \text{ and} \tag{11.1}$$

$$\varphi(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + c_n a(\lambda, \theta, \omega)\lambda^{\frac{n-3}{2}}|x|^{-\frac{n-1}{2}}e^{i\lambda|x|} + O(|x|^{-\frac{n-1}{2}-1}) \tag{11.2}$$

where $\theta = x/|x|$. The function $a \in \mathcal{C}^\infty(\mathbb{R}_\lambda \setminus \{0\} \times \mathbb{S}^{n-1}_\theta \times \mathbb{S}^{n-1}_\omega)$ is called the *scattering amplitude* in the stationary approach. In the course of showing the existence of these plane waves we shall show that a coincides with the scattering amplitude defined in Chapter 10.

The unperturbed plane wave $\exp(i\lambda x \cdot \omega)$ is the value at $\tau = -\lambda$ of the Fourier transform in t of the unperturbed progressing wave:

$$\exp(i\lambda x \cdot \omega) = \int e^{i\lambda t}\delta(t - x \cdot \omega)dt. \tag{11.3}$$

We shall define $\varphi(\lambda, x, \omega)$ in terms of the solution of the continuation problem (1.1). Then, by ???, u is a tempered distribution in the time variable and is, for x in any compact set, smooth for large t. We then define

$$\varphi(\lambda, x, \omega) = \int e^{i\lambda t}u(t, x, \omega)dt. \tag{11.4}$$

THEOREM 11.4. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, if u is the solution of* (1.1) *and φ is defined by* (11.4) *then*

$$\varphi(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + e^{i\lambda|x|}R(\lambda, x, \omega) \tag{11.5}$$

where R has a complete asymptotic expansion as $|x| \to \infty$ on compact sets in $\lambda \in \mathbb{R}\setminus\{0\}$, θ and ω :

$$R(\lambda, x, \omega) \sim c_n\lambda^{\frac{n-3}{2}}a(\lambda, \omega, \theta)|x|^{-\frac{n-1}{2}} + \sum_{j=1}^{\infty}\alpha_j(\lambda, \omega, \theta)|x|^{-\frac{n-1}{2}-j} \quad \text{as } |x| \to \infty; \tag{11.6}$$

here a is the scattering amplitude as defined in (10.38)*, the coefficients $\alpha_j \in \mathcal{C}^\infty(\mathbb{R}_\lambda \times \mathbb{S}^{n-1}_\omega \times \mathbb{S}^{n-1}_\theta)$ and c_n is the universal constant*

$$c_n = . \tag{11.7}$$

PROOF. With u, as in (8.79), the solution of the continuation problem let

$$w = u - \delta(t - x\cdot\omega) \tag{11.8}$$

be the perturbation of the free wave which results. Thus w satisfies

$$P_V w = -V(x)\delta(t - x\cdot\omega),\ w = 0 \text{ for } t < -\rho. \tag{11.9}$$

Recall that the forward fundamental solution for P_V, as in Theorem 6.2, satisfies

$$\operatorname{supp} E \subseteq \{(t, x, x') \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n; t \geq |x - x'|\}. \tag{11.10}$$

Since we can then write

$$w = E * (-V(x)\delta(t - x\cdot\omega)) \tag{11.11}$$

where the convolution is in the time variable, i.e. formally

$$w = -\int E(t-s, x, x')V(x')\delta(s - x'\cdot\omega)ds\,dx' \tag{11.12}$$

it follows that

$$w = 0 \text{ for } |x| > t + \rho. \tag{11.13}$$

Thus w has compact support in the x-variable and we can apply the modified Radon transform of Lax and Phillips:

$$\begin{pmatrix} w \\ D_t w\end{pmatrix} = \mathcal{R}^{-1}(\alpha(t, s, \omega, \theta) - \delta(t-s)\delta_\theta(\omega)). \tag{11.14}$$

Written out this gives

$$\begin{aligned} w &= 2^{-\frac{1}{2}}(\pi)^{-\frac{n-1}{2}}(R^t D_s^{\frac{n-3}{2}}(\alpha(t, s, \omega, d\theta) - \delta(t-s)\delta_\theta(\omega)) \\ &= 2^{-\frac{1}{2}}\pi^{-\frac{n-1}{2}}\int_{\mathbb{S}^{n-1}} k(t, x\cdot\theta, \theta)d\theta - e^{i\lambda x\cdot\omega}\end{aligned} \tag{11.15}$$

where $k(t, s, \omega, \theta) = D_s^{\frac{n-3}{2}}\alpha(t, s, \omega, \theta) - \delta(t-s)\delta_\omega(\omega))$ satisfies

$$\begin{aligned} (D_t + D_s + V_{\mathrm{LP}})k &= 0 \\ k = D_s^{\frac{n-3}{2}}\delta(t-s)\delta_\theta(\omega),&\ t < -\rho.\end{aligned} \tag{11.16}$$

The structure of k follows from the disussion in Chapter 10. In particular, for s bounded k has a complete asymptotic expansion as $t \to \infty$ and in $s > \rho$ it is a function only of $t-s$, ω and θ. Furthermore it is singular on on $t=s$ and $t+s=0\cap\{|s|\le\rho\}$. It follows that the t-Fourier transform of k :

$$\hat{k}(\tau,s,\omega,\theta)=\int e^{-i\tau t}k(t,s,\omega,\theta) \tag{11.17}$$

is a meromorphic function of τ with poles of finite rank only at the points $\tau=-\lambda_j$ where λ_j is an eigenvalue of the infinitesmal generator of the Lax-Phillips semigroup. Furthermore $\hat{\ell}=\hat{k}-(-\tau)^{\frac{n-3}{2}}e^{is\tau}\delta_\theta(\omega)$ is smooth in all variables.

From (11.15) and (11.4)

$$\varphi(\lambda,x,\theta)=e^{i\lambda x\cdot\theta}+\int\limits_{\mathbb{S}^{n-1}}\hat{\ell}(-\lambda,x\cdot\omega,\omega,\theta)d\omega. \tag{11.18}$$

Set $x=|x|\theta'$ with $\theta'\in\mathbb{S}^{n-1}$ fixed. The properties of k are reflected in the fact that

$$\begin{gathered}\ell(t,s,\omega,\theta)=0 \text{ in } s<\rho a\\ \ell(t,s,\omega,\theta)=\ell'(t-s,\omega,\theta) \text{ in } s>\rho a\end{gathered} \tag{11.19}$$

where $\ell'(t,\omega,\theta)$ is supported in $t>-2\rho$, is conormal at $t=0$, $\omega=\theta$ and has a full expansion in exponential as $t\to\infty$. Moreover the wavefront set of ℓ' does not meet the conormal to $t=0$ so

$$\begin{gathered}\hat{\ell}(-\lambda,s,\omega,\theta)=e^{i\lambda s}\mu(\lambda,s,\omega,\theta) \text{ where}\\ \mu=0 \text{ in } s<-\rho,\ D_s\mu=0 \text{ in } s>\rho\end{gathered} \tag{11.20}$$

and μ is $\mathcal{C}^\infty$ in s, ω and θ as a meromorphic function of λ.

We proceed to investigate the structure of

$$\varphi(\lambda,x,\theta)=e^{i\lambda x\cdot\theta}+\int\limits_{\mathbb{S}^{n-1}}e^{i|x|\theta'\cdot\omega}\mu(\lambda,|x|\theta'\cdot\omega,\omega,\theta)d\omega \tag{11.21}$$

by using the stationary phase lemma. The only critical points of the phase function $\lambda|x|\theta'\cdot\theta$ are at $\theta'=\pm\omega$, at which points it takes the values $\pm\lambda|x|$. The amplitude μ vanishes identically near $s=-|x|$, for $|x|>\rho$, and $\mu(\lambda,|x|\theta'\cdot\omega,\omega,\theta)$ is a symbol in $|x|$ near $\theta'=\omega$ since it is actually independent of $|x|$ there. Thus inserting a smooth cutoff $\phi(\theta',\omega)$ with support near $\theta'=\pm\omega$ gives

$$\begin{aligned}\varphi'(\lambda,|x|\theta',\theta)&=\int\phi(\theta',\omega)e^{i|x|\theta'\cdot\omega}\mu(\lambda,|x|\theta'\cdot\omega\omega,\theta)d\omega\\ &\sim e^{i|x|\lambda}\sum_{j=0}^{\infty}a_j(\lambda,\theta',\theta)|x|^{-\frac{n-1}{2}}\end{aligned} \tag{11.22}$$

where

$$\begin{gathered}\varphi(\lambda,|x|\theta',\theta)=\varphi'(\lambda,|x|\theta',\theta)+\varphi''(\lambda,|x|\theta',\theta)\\ \varphi''(\lambda,|x|\theta',\theta)=\int\limits_{\mathbb{S}^{n-1}}(1-\phi)(\theta',\omega)e^{i\lambda|x|\theta'\cdot\omega}\mu(-\lambda,x\cdot\omega,\omega,\theta)d\omega.\end{gathered} \tag{11.23}$$

Here the cufoff factor constrains the support of the integrand to some set $|\theta'\cdot\omega|\le r$, for $r<1$.

It remains to show that φ'' is rapidly decreasing as $|x|\to\infty$. Since $|\theta'\cdot\omega|\le r<1$ on the support of the integrand in (11.23) the decomposition as in (2.57) can be used:

$$\omega = t\theta' + (1-t^2)^{\frac{1}{2}}\omega',\ \omega' \perp \theta' \tag{11.24}$$

and then $s = t/|x|$ and ω' can be used as variables of integration:

$$\begin{gathered}\varphi''(\lambda, |x|\theta', \theta) = \\ |x|^{-1} \int_{(\theta')^\perp} \int_{-\rho\le|s|\le r|x|} (1-\phi)(\theta', |x|^{-1}s\theta' + (1-|x|^{-2}s^2)^{\frac{1}{2}}\omega') \\ e^{is\lambda}\mu(-\lambda, s, \||x|^{-1}s\theta' + (1-|x|^{-2}s^2)^{\frac{1}{2}}\omega', \theta)(1-|x|^{-2}s^2)^{\frac{n-3}{2}} ds d\omega'.\end{gathered} \tag{11.25}$$

Since $\lambda \neq 0$ the identity $d\exp(is\lambda)/ds = i\lambda\exp(is\lambda)$ and integration by parts can be used to write this in the form

$$\begin{gathered}\varphi''(\lambda, |x|\theta', \theta) = \\ (-i\lambda|x|)^{-1} \int_{(\theta')^\perp} \int_{-\rho\le|s|\le r|x|} e^{is\lambda}\frac{d}{ds}\mu(-\lambda, s, |x|^{-1}s\theta' + (1-|x|^{-2}s^2)^{\frac{1}{2}}\omega', \theta) \\ (1-|x|^{-2}s^2)^{\frac{n-3}{2}} ds d\omega' + \widetilde{\varphi}.\end{gathered} \tag{11.26}$$

Here we have used the fact that $\phi \equiv 0$ on the support of $ds\mu/ds$. The remainder term in (11.26), $\widetilde{\varphi}$, is given by a and integral as in (11.25) with $(1-\phi)$ replaced by its s-derivative. The support is therefore in $r|x| \ge |s| > \frac{1}{2}r|x|$ and the stationary phase lemma shows it to be rapdily decreasing.

Finally then the asymptotic behaviour of the integral in (11.26) needs to be analyzed. It is clearly a smooth function of $|x|^{-1}$ as $|x| \to \infty$ and the coefficients of its Taylor series, i.e. asymptotic expansion in $|x|$ are linear combinations of the functions

$$\int_{(\theta')^\perp} \int s^p e^{is\lambda} \frac{d}{ds} s^k V^q_{\omega'}\mu(-\lambda, s, \omega', \theta) ds d\omega' \tag{11.27}$$

where $q \le k$ and V'_ω is the unit vector field on the circle $t\omega' + (1-t^2)^{\frac{1}{2}}[\omega'$ for each $\omega' \in (\theta')^\perp$. Integrating by parts in s again allowss the terms in (11.27) to be expressed in terms of the integrals

$$\int_{(\theta')^\perp} \int s^p V^q_{\omega'}\hat{\ell}(-\lambda, s, \omega', \theta) ds d\omega',\ q \le p. \tag{11.28}$$

That these all vanish follows from Theorem 2.8 and the compactness of the support of w for each t. □

In fact it is easy to see directly that the terms in the expansion of (11.26) must all vanish. Without using Theorem 2.8 we have shown that $\varphi(\lambda, x, \theta)$ satisfies

$$\begin{gathered}(\Delta + V - \lambda^2)\varphi = 0 \\ \varphi \sim e^{i\lambda x\cdot\theta} + e^{i\lambda|x|}R(\lambda, x, \theta) + R'(\lambda, x, \theta)\end{gathered} \tag{11.29}$$

where both R and R' have complete asymptotic expansions as $|x| \to \infty$. It follows that each terms must satisfy the same equation asymptotically, in particular

$$(\Delta + V - \lambda^2)R'(\lambda, x, \theta) = O(|x|^{-\infty}) \text{ as } |x| \to \infty. \tag{11.30}$$

Writing Δ in polar coordinates and computing the leading part it can be seen that (11.30) implies that R' is itself rapidly decreasing as $|x| \to \infty$.

The generalized outgoing eigenfunctions as in (11.5) can be used to give an explict spectral representation for $\Delta + V$ similar to that for Δ given by the Fourier inversion formula:

$$\Delta = \int_0^\infty \lambda dE_0(\lambda) \tag{11.31}$$

where the spectral projection $E_0(\lambda)$ is given by

$$E_0(\lambda)f(x) = \int_{\mathbb{S}^{n-1}} \lambda^{\frac{n-1}{2}} \varphi_0(x, \sqrt{\lambda}, \omega) \lambda^{\frac{n-1}{2}} \varphi_0(y, \sqrt{\lambda}, \omega) d\omega \tag{11.32}$$

with the

$$\varphi_0(x, \lambda, \omega) = e^{i\lambda x \cdot \omega} \tag{11.33}$$

being generalized eigenfunctions for Δ. We shall find a similar spectral resolution for $\Delta + V(x)$ using the generalized eigenfunctions φ in place of φ_0.

The spectrum of $\Delta + V$, for $V \in \mathcal{C}_c^\infty(\mathbb{R})$ real valued, differs little from that of for Δ. Namely, the absolutely continuous spectrum for $\Delta + V$ is $[0, \infty)$, as in the unperturbed case, and there are may in addition be a finite number of negative eigenvalues. We proceed to prove this, for $n \geq 3$ odd.

First we proceed to show that $\Delta + V$, with V real-valued, is a selfadjoint operator with domain $\operatorname{Dom}(\Delta + V) = H^2(\mathbb{R}^n)$.

The resolvent is well defined for $\Im\sigma < 0$, $(\Delta - \sigma)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ and also the operator

$$B_\sigma = V(x)(\Delta - \sigma)^{-1}. \tag{11.34}$$

The reason to consider B_σ is that the existence of negative eigenvalues is very much tied up with the invertability of the operator

$$Id + B_\lambda, \ \lambda < 0. \tag{11.35}$$

We shall prove that $Id + B_{\sigma^2}$ is a Fredholm operator depending analytically on σ. Then we shall use the analytic Fredholm theorem to conclude that the set of σ's forr which $Id + B_{\sigma^2}$ is not invertible is discrete in $\mathbb{C}$.

Proposition 11.28. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, B_{σ^2} is compact for $\Im\sigma^2 < 0$.*

Proof. $(\Delta - \sigma^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ is bounded for $\Im\sigma < 0$. Since V has compact support, $\operatorname{supp} V \subset B(0, \rho)$, then

$$V(x)(\Delta - \sigma^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(B(0, \rho)) \tag{11.36}$$

is bounded. By Rellich's compactness theorem the inclusion

$$H^2(B(0, \rho)) \hookrightarrow L^2(\mathbb{R}^n) \tag{11.37}$$

is compact, proving the proposition. □

Theorem 11.5. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $n \geq 3$ odd,*

$$(\Delta - \sigma^2)^{-1} : H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \ \Im\sigma << 0 \tag{11.38}$$

has a meromophic extension to $\mathbb{C}$ as an operator

$$(\Delta - \sigma^2)^{-1} : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n). \tag{11.39}$$

Proof. First we shall check that $C_{\sigma^2} = (\Delta - \sigma^2)^{-1}$ is weakly holomorphic. That is we proceed to show that

$$g_\sigma = (C_{\sigma^2} f, \varphi) = \langle (\Delta - \sigma^2)^{-1} f, \varphi \rangle \tag{11.40}$$

has a meromorphic extension to $\mathbb{C}$ for $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Clearly g_σ is analytic for $\Im\sigma << 0$. Also g_σ makes sense for $\Im\sigma > 0$. The problem is then to analytically continue g_σ across the real axis. We have

$$g_\sigma = \int \frac{\hat{f}(\xi)\hat{\varphi}(-\xi)}{|\xi|^2 - \sigma^2}, \ \Im\sigma << 0. \tag{11.41}$$

Changing to polar coordinates we obtain

$$g_\sigma = \int_{\mathbb{S}^{n-1}} \int_0^\infty r^{n-1} \frac{\hat{f}(r\omega)\hat{\varphi}(-r\omega)}{r^2 - \sigma^2} dr\, d\omega, \Im\sigma < 0 \tag{11.42}$$

We will first extend g_σ analytically to $\mathbb{C} \setminus (-\infty, 0]$ by using the Cauchy integral formula. For $\Im\sigma < 0$ we have using the contour of Figure***

$$\begin{aligned} g_\sigma = & \int_{S^{n-1}} \int_0^{i\delta} \frac{\zeta^{n-1}\hat{f}(\zeta w)\hat{\varphi}(-\zeta w) d\zeta d\omega}{\zeta^2 - \sigma^2} \\ & + \int_{S^{n-1}} \int_{\Re\zeta > 0, \Im\zeta = \delta} \frac{\zeta^{n-1}\hat{f}(\zeta\omega)\hat{\varphi}(-\zeta\omega)}{\zeta^2 - \sigma^2} d\zeta d\omega \\ & + \pi i \int_{S^{n-1}} \sigma^{n-2} \hat{f}(\sigma\omega)\hat{\varphi}(-\sigma\omega) d\omega. \end{aligned} \tag{11.43}$$

Formula (11.43) clearly implies that g_σ can be extended to $\mathbb{C} \smallsetminus (-\infty, 0]$. This is valid in any dimension. Next we define $g'_\sigma = g_{-\sigma}$ for $\Im\sigma > 0$. We analytically continue g'_σ across $(-\infty, 0]$ using an analogous formula to (11.43).

However, the extension of g_σ to the upper half plane differs from g'_σ in the following fashion

$$g_\sigma = g'_\sigma + p_\sigma, \quad \Im\sigma > 0, \tag{11.44}$$

where

$$p_\sigma = \pi i (-\sigma)^{n-2} \int_{S^{n-1}} \hat{f}(-\sigma\omega)\hat{\varphi}(\sigma\omega) d\omega \tag{11.45}$$

If we continue analytically g'_σ as indicated above we get

$$g'_\sigma = g_\sigma + p_\sigma + p_{-\sigma} \quad \text{in } \Im\sigma < 0. \tag{11.46}$$

However, for dimension n odd it is easy to check that $p_{-\sigma} = -p_\sigma$, concluding that

$$g'_\sigma = g_\sigma \text{ in } \Im\sigma < 0 \tag{11.47}$$

and obtaining the desired extension of g_σ to $\mathbb{C} - \{0\}$. The last thing to check is that g_σ has a removable singularity at $\sigma = 0$. For that we will prove that g_σ is bounded near $\sigma = 0$. Using (11.42) and the fact that

$$\frac{2r}{r^2 - \sigma^2} = \frac{d}{dr} \log(r^2 - \sigma^2) \quad \Im\sigma < 0 \tag{11.48}$$

we conclude

$$g_\sigma = \frac{1}{2}\int_0^\infty \int_{S^{n-1}} r^{n-2}\frac{d}{dr}(\log(r^2-\sigma^2))\hat{f}(r\omega)\hat{\varphi}(-r\omega)d\omega\,dr. \tag{11.49}$$

Integrating by parts in the r-variable we get

$$g_\sigma = -\frac{1}{2}\int_0^\infty \int_{S^{n-1}} \frac{d}{dr}(r^{n-2}\hat{f}(r\omega)\hat{\varphi}(-r\omega))\log(r^2-\sigma^2)d\omega\,dr \tag{11.50}$$

For $n \geq 3$, (11.50) is clearly bounded near $\sigma = 0$. □

Using the analytic Fredholm theorem then we conclude that there is at most a discrete numbers of σ's for which $(Id + B_{\sigma^2})$ is not invertible. We will use this to prove

Proposition 11.29. *Let $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ odd and V real-valued. Then $\Delta + V$ has at most a discrete number of negative eigenvalues.*

Proof. Let $\{\lambda_i\}$ be eigenvalues of $(\Delta + V)$, $\lambda_i < 0$ with

$$(\Delta + V)\varphi_i = \lambda_i\varphi_i, \quad \varphi_i \in H^2(\mathbb{R}^n) \tag{11.51}$$

Now we define $f_i \varepsilon L^2(\mathbb{R}^n)$ by

$$f_i = (\Delta - \lambda_i)\varphi_i \tag{11.52}$$

and consequently since $\lambda_i < 0$, $\varphi_i = (\Delta - \lambda_i)^{-1}f_i$. From (11.51) we then conclude

$$f_i + V(x)(\Delta - \lambda_i)^{-1}f_i = 0, \tag{11.53}$$

that is

$$(Id + B_{\sqrt{\lambda_i}})f_i = 0. \tag{11.54}$$

Now the theorem follows from the fact that there are at most a discrete set of σ's for which there is a non-trivial kernel of $(Id + B_\sigma)$. We now state □

Theorem 11.6. *Let $V \in C_0^\infty(\mathbb{R}^n)$ with V real-valued and $n \geq 3$ odd. Then there is at most a finite number of eigenvalues of $\Delta + V$.*

Proof. The only remaining thing to check is that the spectrum (set of eigenvalues) of $\Delta + V$ is bounded. Let λ be an eigenvalue for $(\Delta + V)$, then

$$(\Delta + V)f = \lambda f, \quad f \in H^2(\mathbb{R}^n), f \neq 0. \tag{11.55}$$

Then

$$\langle(\Delta + V)f, f\rangle = \langle\lambda f, f\rangle \tag{11.56}$$

where $\langle\ ,\ \rangle$ denotes the L^2 inner product.

Since V is bounded we conclude

$$\int |\nabla f|^2 + \int V|f|^2 = \lambda \int |f|^2. \tag{11.57}$$

Therefore

$$\int (V(x) - \lambda)|f|^2 \leq 0 \tag{11.58}$$

and consequently $\lambda \geq \inf V(x)$

In a similar fashion we conclude $\lambda \leq \sup V(x)$.

The possible existence of negative eigenvalues for $\Delta + V$ (the so called bound states of V) implies that there might be solutions of the wave equation plus potential

$$(\frac{\partial^2}{\partial t^2} - (\Delta + V))u = 0 \tag{11.59}$$

of the form

$$u = e^{i\lambda_j t} w_j(x) \tag{11.60}$$

where $-\lambda_j^2$ is an eigenvalue of $(\Delta + V)$ and w_j, a corresponding eigenfunction. Intuitively speaking one cannot "scatter" these solutions since in any compact set the energy of the solutions does not decay in time. In order to do "scattering" in the classical sense we have to avoid solution of the form (11.60).

We shall prove next that if $V \in C_0^\infty(\mathbb{R}^n)$ n odd and with no bound states, the local energy of solutions of $P_V u = 0$ decays for large time. To do this we shall use the Lax-Phillips semigroup introduced in chapter 10. We proved there that the Lax-Phillips semigroup $Z(t) : L^2((-\rho,\rho) \times S^{n-1}) \to L^2((-\rho,\rho) \times S^{n-1})$ is compact for $t > 2\rho$ where $\operatorname{supp} V \subseteq B(0,\rho)$. Let

$$\widetilde{K} = \mathcal{R}^{-1}(L^2(-\rho,\rho) \times S^{n-1}) \tag{11.61}$$

provided with the norm induced by the energy norm and $\mathcal{R} = LP$. Then

$$\widetilde{Z}(t) = \mathcal{R}^{-1} Z(t) \mathcal{R} : \widetilde{K} \to \widetilde{K} \tag{11.62}$$

is compact for $t > 2\rho$. $\widetilde{Z}(t)$ is the transformed Lax-Phillips semigroup and an analogous statement to Proposition 10.24 holds for $\widetilde{Z}(t)$. □

Proposition 11.30. *$\|\widetilde{Z}(t)\| < 1$, $t \geq 0$ if V has no bound states, where $\| \quad \|$ denotes the operator norm.*

Proof. We know already that $\|\widetilde{Z}(t)\| \leq 1$ $\forall t \geq 0$, so that it is enough to show that there is not eigenvalue of $\widetilde{Z}(t), t \geq 0$ with norm 1. Suppose the opposite. Let $\lambda \in \mathbb{C}$ eigenvalue of $\widetilde{Z}(T)$ with $|\lambda| = 1$. Without less of generality (since $\widetilde{Z}(t)$ is a semigroup) we can assume $T = 3\rho$. Let N be the null space of $\widetilde{Z}(T) - \lambda \operatorname{Id}$. Since $\widetilde{Z}(T) - \lambda \operatorname{Id}$ is Fredholm, N is finite dimensional. Since $\widetilde{Z}(t)$ commutes with $\widetilde{Z}(T)$ for t positive

$$\widetilde{Z}(t) : N \to N. \tag{11.63}$$

Let B be the infinitesimal generator of $\widetilde{Z}(t)$ on B, i.e., $\widetilde{Z}(t) = e^{Bt}$.

There is an eigenvalue $\mu \in \mathbb{C}$ of B with $\Re\mu = 0$ such that

$$e^{\mu T} = \lambda. \tag{11.64}$$

Let w be an eigenfunction of B associated to the eigenvalue μ. Then $Bw = \mu w$ and therefore

$$\widetilde{Z}(t) w = e^{\mu t} w. \tag{11.65}$$

Since $Z(t)$ is smoothing for large t, then $w \in C^\infty$.

From the definition of $\widetilde{Z}(t)$ we have

$$\widetilde{Z}(t) w = \mathcal{R}^{-1} Z(t) \mathcal{R} w \tag{11.66}$$

where $Z(t)$ is defined as in (10.7).

$\mathcal{R}$ is unitary and since $\|\widetilde{Z}(t)w\| = \|w\|$ we conclude since the restriction operators used to define $Z(t)$ have norm less or equal than one that

$$\widetilde{Z}(t)w = \mathcal{R}^{-1}W_V(t)\mathcal{R}w = e^{\mu t}w \tag{11.67}$$

and consequently

$$U_V(t)w = e^{\mu t}w. \tag{11.68}$$

$U_V(t)$ is a unitary group and it is easy to check that its infinitesimal generator is given by

$$A = \begin{pmatrix} 0 & Id \\ \Delta + V & 0 \end{pmatrix} \tag{11.69}$$

We have that $w \in D(A)$, since w is smooth. From (11.68) we conclude then

$$Aw = \mu w. \tag{11.70}$$

and using (11.69)

$$(\Delta + V - \mu^2)w_2 = 0 \text{ where } w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{11.71}$$

and moreover $w_2 \in L^2(\mathbb{R}^n)$ since w has finite energy.

Formula (11.71) contradicts the assumption of no bound states for V. □

THEOREM 11.7. *Suppose $V \in C_0^\infty(\mathbb{R}^n)$ has no bound states then*

$$\begin{aligned} \|Z(t)\| &\le Ce^{-\beta t} \\ \|\widetilde{Z}(t)\| &\le Ce^{-\beta t} \end{aligned} \tag{11.72}$$

for some $C > 0$ and $\beta > 0$.

PROOF. This is a standard result from the theory of semigroups which follows using the asymptotic decaying exponentials (since there are no eigenvalues of B with 0 real part) proved in (10.17). □

An immediate consequence of the exponential decay of the Lax-Phillips semigroup are the following which shall be used later.

PROPOSITION 11.31. *Let $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, for $n \ge 3$, be real-valued with no bound states, then*

$$\int_{-\infty}^{a} \int |W_V(t)k(s,w)|^2 d\omega\, ds \underset{t\to\infty}{\rightarrow} 0, \quad \forall k \in L^2(\mathbb{R} \times S^2). \tag{11.73}$$

Let u be as in (1.1)*, then $\exists C > 0$, $B > 0$ such that*

$$\sum_{\substack{x \in K \subset\subset \mathbb{R}^n \\ \omega \in \mathbb{S}^{n-1}}} |u(t,x,w)| \le Ce^{-\beta t} \tag{11.74}$$

for t sufficiently large. Let $\alpha(t,s,\theta,w)$ be as in Proposition 8.20, then $\exists \beta > 0$, $C > 0$ such that

$$\sup_{\substack{s \in K \subset\subset \mathbb{R}^n \\ \theta, w \in \mathbb{S}^{n-1}}} |\alpha(t,s,\theta,w)| \le Ce^{-\beta t} \tag{11.75}$$

for t sufficiently large.

We shall use this fundamental result in the development of the scattering theory of Lax-Phillips.

The first objective in this theory is to prove that a solution of the equation

$$P_V u = 0 \tag{11.76}$$

with finite energy, there exist solutions $u_\pm$ of the unperturbed wave equation ($V = 0$) such that

$$\lim_{t\to\pm\infty} \|(u, \frac{\partial u}{\partial t}) - (u_\pm, \frac{\partial u_\pm}{\partial t})\|_{Ho} = 0 \tag{11.77}$$

The "wave operators" are defined by

$$\widetilde{W}_\pm u = u_\pm \tag{11.78}$$

with u, $u_\pm$ as in (11.76), (11.77), and the Scattering operator (if it exists) as defined by

$$\widetilde{S} = \widetilde{W}_+ \widetilde{W}_-^{-1}. \tag{11.79}$$

To do this we shall work rather in Radon transform land

THEOREM 11.8. *Let $n \geq 3$ odd, $V \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} V \subseteq B(0,\rho)$, V real valued. Then we can decompose*

$$L^2(\mathbb{R} \times S^{n-1}) = D_- \oplus K \oplus D_+ \tag{11.80}$$

orthogonal sum with respect to $\| \quad \|_V$ defined by

$$\|f\|_v = \|R^{-1} f\|_{H_V}, \tag{11.81}$$

where

$$\begin{aligned} D_- &= L^2((-\infty,-\rho) \times S^{n-1}) \\ D_+ &= L^2((\rho,\infty) \times S^{n-1}) \\ K &= L^2((-\rho,\rho) \times S^{n-1}) \end{aligned} \tag{11.82}$$

D_+ is called the outgoing space, D_- the incoming one and K the interaction space.

PROOF. We first show that the scalar product

$$\langle f, g\rangle_V = \langle \mathcal{R}^{-1} f, \mathcal{R}^{-1} g\rangle_{H_V} \tag{11.83}$$

induced by $\| \quad \|_V$ coincides with the standard L^2-inner product on $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$, i.e.

$$\langle f, g\rangle_V = \langle f, g\rangle_0, \quad f, g in D_+ \text{ or } f, g \in D_- \tag{11.84}$$

This can be readily seen by observing that

$$\begin{gathered} (\operatorname{supp} \mathcal{R}^{-1} f) \cap \operatorname{supp} V = \phi \text{ for } f \in D_+ \cup D_- \text{ and } \|u\|_{H_V} = \|u\|_{H_0} \\ \text{if } \operatorname{supp} u \cap \operatorname{supp} V = \emptyset, \end{gathered} \tag{11.85}$$

since $f = 0$ for $|s| \leq \rho$ implies $R^t f = 0$ for $|x| \leq \rho$. Therefore we conclude $D_+ \perp D_-$. To prove that $K \perp (D_- \oplus D_+)$, we observe for $f \in K$, $g \in D_\pm$

$$\langle f, g\rangle_V = \langle \mathcal{R}^{-1} f, \mathcal{R}^{-1} g\rangle_{H_V} = \langle \mathcal{R}^{-1} f, \mathcal{R}^{-1} g\rangle_{H_0} \tag{11.86}$$

again using (11.85).

Now by the unitary of the modified Radon transform of Lax and Phillips

$$\langle \mathcal{R}^{-1} f, \mathcal{R}^{-1} g\rangle_{H_0} = \langle f, g\rangle_{L^2(\mathbb{R})} = 0 \tag{11.87}$$

proving the proposition. □

The fact that the energy of solutions of $P_V u = 0$ remaining near the interaction space is very small for large time if V has no bound states is exploited to prove the existence of the so called wave operators $W_\pm$ and the scattering operator.

THEOREM 11.9. *Let $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ n odd and V with no bound states. Then*

$$W_\pm k = \lim_{t\to\pm\infty} T_{-t}W(t)k \tag{11.88}$$

defines an unitary isomorphism

$$W_\pm : (L^2(\mathbb{R}\times S^{n-1}), \| \quad \|_V) \to (L^2(\mathbb{R}\times S^{n-1}), \| \quad \|_0) \tag{11.89}$$

where T_t is the group of translations to the right

$$T_t f(s,\omega) = f(s-t,\omega), f \in L^2(\mathbb{R}\times S^{n-1}). \tag{11.90}$$

PROOF. We shall give the proof for W_+. An analogous argument will give the result for W_-.

We shall show first that W_+ is an isometry defined on

$$E_+ = \{\in L^2(\mathbb{R}\times S^{n-1}); W_V(t)k \in D_+ \text{ for some } t\} \tag{11.91}$$

We shall prove later that E_+ is dense in $L^2(\mathbb{R}\times S^{n-1})$.

Let $k \in E_+$, then $W_V(t_0)k \in D_+$. Using again (11.85) and the fact that $\mathcal{R}U_0(t) = T_t\mathbb{R}$ we have that

$$W_V(t)f = T_t f \quad \text{for } t \geq 0(\text{ resp. } t \leq 0) f \in D_+ \text{ (resp. } f \in D_-). \tag{11.92}$$

Then

$$W_V(t)k = W_V(t-t_0)W_V(t_0)k = T_{(t-t_0)}W_V(t_0)k, \quad t \geq t_0 \tag{11.93}$$

then

$$T_{-t}W_V(t)k = T_{-t}T_{t-t_0}W_V(t_0)k = T_{-t_0}W_V(t_0)k, t \geq t_0. \tag{11.94}$$

Thus we conclude that

$$W_+k = T_{-t_0}W_V(t_0)k, \quad k \in E_+ \tag{11.95}$$

and

$$\|W_+k\|_{L^2(\mathbb{R}\times S^{n-1})} = \|T_{-t_0}W_V(t_0)k\|_{L^2(\mathbb{R}\times S^{n-1})} = \|W_V(t_0)k\|_{L^2(\mathbb{R}\times S^{N-1}}. \tag{11.96}$$

Using (11.84) we obtain that W_t is an isometry on E_+, namely

$$\|W_+k\|_{L^2(\mathbb{R}\times S^{n-1})} = \|W_V(t_0)k\|_{L^2(\mathbb{R}\times S^{n-1})} = \|k\|_V. \tag{11.97}$$

Next we shall check that E_+ is dense in $L^2(\mathbb{R}\times S^{-1})$. Let $\widetilde{k} \in (E_+)^\perp$. Then

$$\langle \widetilde{k}, \ell\rangle_{L^2(\mathbb{R}\times S^{n-1})} = 0 \quad \forall \tag{11.98}$$

Assume $W_V(t_0)\ell \in D_+$. By the local energy decay of solutions of $P_V u = 0$ we have that given $\varepsilon > 0$, for t sufficiently large

$$\int_{-\infty}^{\rho}\int_{S^{n-1}} |W(t)\widetilde{k}\|^2 dw\, ds < \varepsilon. \tag{11.99}$$

Because of (11.85) and the unitary of $\mathcal{R}$, the norms $\| \quad \|_V$ and $\| \quad \|_0$ are equivalent for $n \geq 3$ and therefore

$$\|W_V(t)\widetilde{k}\|_V \leq C\|W_V(t)\widetilde{k}\|_{L^2} \leq C\varepsilon \tag{11.100}$$

for t sufficiently large and since

$$\|W_V(t)\widetilde{k}\|_V = \|\widetilde{k}\|_V \forall t \tag{11.101}$$

we get finally $\widetilde{k} = 0$.

We have proven so far then, that

$$W_+ : (L^2, \| \quad \|_V) \to (L^2, \| \quad \|_0) \tag{11.102}$$

is an isometry. To prove the isomorphism we need to check that W_+ is onto. Since W_+ is an isometry (and therefore has closed range) is enough to check that the range of W_+ is dense. This follows since we shall prove that

$$\operatorname{Ran} W_+ \subseteq \langle \ell \in L^2(\mathbb{R} \times S^{n-1});\ \ell = 0, \tag{11.103}$$

and A is dense in $L^2(\mathbb{R} \times S^{n-1})$. To see this we take $\ell \in A$. Then

$$T_t \ell \in D_+ \text{ for } t \geq \rho - c. \tag{11.104}$$

Therefore

$$W_V(-t)T_t\ell \in E_+ \text{ for } t \geq \rho - c \tag{11.105}$$

and

$$W_+(W_V(-t)T_t\ell) = T_{-t}W_V(t)T_t\ell = \ell. \tag{11.106}$$

We leave as an exercise to the reader to check the unitarity of the W_+. □

The scattering operator is then defined by

$$S = W_+ W_-^{-1}. \tag{11.107}$$

We check now that the kernel of the scattering operator is the scattering kernel defined in section 8.

THEOREM 11.10. *Let $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ odd with no bound states then S is an unitary isomorphism*

$$S : (L^2(\mathbb{R} \times S^{n-1}), \| \quad \|_0) \to (L^2(\mathbb{R} \times S^{n-1}), \| \quad \|_0). \tag{11.108}$$

Moreover the Schwarz kernel of S is given by

$$K_S(s, \theta, \overline{s}, \omega) = K_S(s, \overline{s}, \theta, \omega) = \mathcal{K}_V(s - \overline{s}, \theta, \omega) \tag{11.109}$$

where $\mathcal{K}_V$ is the scattering kernel as defined in (8.72).

PROOF. The fact that S is an unitary isomorphism follows directly from Theorem 11.9. Formula (11.109) follows from the following relation for the kernel of the scattering operator which is based on finite propagation speed of solutions of $P_V u = 0$

$$K_S(s, \theta, \overline{s}, \omega) = (T_{-t}W_V(t + r)T_{-t}(\delta_{\overline{s}}(\cdot)\delta_\omega(\cdot))(s, \theta) \tag{11.110}$$

for $t > \overline{s} + \rho, s + r > \rho$. Of course, (11.110) is meant in the sense of distributions and we have extended the operators to spaces of distributions.

Let

$$u = \delta_{\overline{s}}(\cdot)\delta_\omega(\cdot). \tag{11.111}$$

We have that

$$W_V(t')T_{-t'}u = W_V(t)W_V(t'-t)T_{-t'}u \tag{11.112}$$

Now

$$W_V(t'-t)T_{-t'}u = T_{t'-t}T_{-t'}u = T_{-t}u, \quad t' \leq t \tag{11.113}$$

since $\operatorname{supp} T_{-t'}u \subseteq \{(s,\omega); s < -\rho\}$. Now

$$W_v(t'-t)f = T_{t'-t}f, t' \leq t \text{ if } \operatorname{supp} f \subset \{s \leq \bar{s}\}. \tag{11.114}$$

Now let $\phi_j \in C_0^\infty(\mathbb{R} \times S^{n-1})$ an approximation to u (depending smoothly on $\bar{s}, \omega$). We have

$$T_{-r'}W_V(t+r')T_{-t}\phi_j(s,\theta) = \tag{11.115}$$

$$T_{-r}T_{r-r'}W_V(-r+r')W_V(r)W_V(t)T_{-t}\phi_j)(s,\theta) = \tag{11.116}$$

$$T_{r-r'}W_V(r'-r)W_V(r)W_V(t)T_{-t}\phi_j(r+s,\theta). \tag{11.117}$$

Since $r+s > \rho$ then $r'+s > \rho$ for $r' \geq r$ and $t > \bar{s}+\rho$.

Now using the domain of dependence property (6.4) we check that

$$W_V(r'-r)W(r+t)T_{-t}\phi_j(r'+s,\theta) = T_{r'-r} \tag{11.118}$$

for j sufficiently large.

Then we conclude from (11.115) and (11.118)

$$T_{-r'}W_V(t+r')T_{-t}\phi_j(s,\theta) = T_{-r}W_V(r+t)T_t\phi_j(s,\theta) \tag{11.119}$$

for j sufficiently large, $t > \bar{s}+\rho$, $s+r > \rho$, proving (11.110).

Now (11.109) follows immediately from the representation (11.110) since the RHS of (11.110) satisfies the equation

$$((D_t - D_s) + V_{\mathrm{LP}})w = 0 \tag{11.120}$$

and the initial conditions match and $s > \rho$ (taking $r = \varepsilon$ small enough), proving the claim. □

From the exponential decay of solutions of $P_V u = 0$ and formula (11.110) we deduce

PROPOSITION 11.32. *Let $V \in C_0^\infty(\mathbb{R}^n)$ real-valued with no bound states, then* $\exists\alpha > 0$, $C > 0$

$$|K_S(s,\theta,w)| \leq Ce^{-\alpha s} \text{ for } s \text{ large.} \tag{11.121}$$

We shall show next that under the assumption of no bound states the wave operators interwine the translation group T_t and the group $W_V(t)$. Since $\mathcal{R}$ is unitary this will imply that the pull back wave operators $\widetilde{W}_V(t) = \mathcal{R}^{-1}W_V(t)\mathcal{R}$ intertwine the free group $U_0(t)$ and the wave group $U_V(t)$.

PROPOSITION 11.33. *Let $V \in C_0^\infty(\mathbb{R}^n)$ real-valued with no bound states. Then*

$$T_tW_\pm = W_\pm W_V(t) \text{ and} \tag{11.122}$$

$$U_0(t)\widetilde{W}_\pm = \widetilde{W}_\pm U_V(t) \text{ where} \tag{11.123}$$

$$\widetilde{W}_\pm = \mathcal{R}^{-1}W_V(t)\mathcal{R}. \tag{11.124}$$

PROOF. It is sufficient to prove (11.122) since $\mathcal{R}$ is unitary.

$$W_\pm = \lim_{t\to\infty} T_{-(r+t)} W_V(t+r) \tag{11.125}$$

for every r. Then

$$W_\pm = T_{-r}(\lim_{t\to\infty} T_{-t} W_V(t)) W_V(r). \tag{11.126}$$

Consequently,

$$T_r W_\pm = W_\pm W_V(r) \text{ for all } r. \tag{11.127}$$

The pull back wave operators $\widetilde{W}_\pm$ and the pull back scattering operator

$$\widetilde{S} = \widetilde{W}_+ \widetilde{W}_-^{-1} \tag{11.128}$$

satisfy analogous property to S. □

THEOREM 11.11. *$\widetilde{W}_\pm$ are unitary isomorphisms*

$$\widetilde{W}_\pm : H_0 \longrightarrow H_V \tag{11.129}$$

and $\widetilde{S}$ is an unitary isomorphism

$$\widetilde{S} : H_0 \longrightarrow H_0 \tag{11.130}$$

The interwining property (11.123) *implies a similar one for the infinitesimal generators of $U_V(t)$ and $U_0(t)$. Thus we conclude*

$$\varDelta \widetilde{W}_\pm^{21} = \widetilde{W}_\pm^{21}(\varDelta + V) \tag{11.131}$$

where $\widetilde{W}_\pm^{21}$ denotes the corresponding component of $\widetilde{W}_\pm$. Thus we conclude that the spectrum of $\varDelta + V$ coincides with the spectrum of $\varDelta$, if V has no bound states.

THEOREM 11.12. *Let $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ n odd, V real-valued with no bound states. Then*

$$\sigma(\varDelta) = \sigma(\varDelta) = \sigma(\varDelta + V) = \sigma_{ac}(\varDelta + V) \tag{11.132}$$

where $\sigma(S)$ denotes spectrum of S and $\sigma_{ac}(S)$ the absolutely continuous spectrum of S.

In the previous discussion we proved that the scattering kernel defined in section 10 coincides with the kernel of the scattering operator as defined via the Lax-Phillips theory if the potential has no bound states.

THEOREM 11.13. *Let $V \varepsilon C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ odd, V with no bounds states, then*

$$f = B_{n-3} f = c_n \int_{S^{n-1}} \int_{\mathbb{R}^n} \int \lambda^{n-3} \varphi(\lambda, x, \omega) \overline{\varphi}(\lambda, x', \omega) f(x') d\lambda dx' d\omega \tag{11.133}$$

$$(\varDelta + V) f = B_{n-1} f = c_n \int_{S^{n-1}} \int_{\mathbb{R}^n} \lambda^{n-1} \varphi(\lambda, x, \omega) \overline{\varphi}(\lambda, x', \omega) f(x') d\lambda dx' d\omega \tag{11.134}$$

with $f \in H^2(\mathbb{R}^n)$ and φ as in (11.5).

Proof. It is enough to check (11.30) since φ satisfies $(\Delta + V)\varphi = \lambda^2\varphi$.

Let $M(t)\ C_0^\infty(\mathbb{R}\times S^{n-1}) \to D'(\mathbb{R}^n)$ be the operator with Schwartz kernel

$$K_{M(t)} = D_t^{\frac{n-3}{2}} u(t-s, x, \omega) \tag{11.135}$$

with u solution of (1.1).

We claim now that M is the first component of the operator

$$\widetilde{M}(t) = U_V(t)\mathcal{R}^{-1}W_-^{-1} \tag{11.136}$$

To check (11.136) we observe that both M and $\widetilde{M}$ satisfy $P_V\widetilde{M} = P_V M = 0$. Then uniqueness of solutions of the IVP for $P_V u = 0$ it is enough to prove if $v_0 = M(t)k|_{t=0}$, $v_1 = \frac{\partial}{\partial t}M(t)k|_{t=0}$, with $k = \delta(s-s')\delta_0(\omega)$

$$(v_0, v_1) = \mathcal{R}^{-1}W_-^{-1}k. \tag{11.137}$$

A similar argument to (11.94) shows that $W_-^{-1}k = T_{-r}W_V(r)k$ for r sufficiently large negative and therefore we get using that

$$\begin{gathered} \mathcal{R}U_0(r) = T_r\mathcal{R} \\ \mathcal{R}^{-1}W^{-1}k = U_0(-r)U_V(r)\mathcal{R}^{-1}k \end{gathered} \tag{11.138}$$

for large r negative. Using the unitarity of $W_\pm$ togethere with the identity $(U_0(-r)U_V(r)) = U_V(-r)U_0(r)$ we conclude that

$$\mathcal{R}^{-1}W_-^{-1}k = U_V(-r)\mathcal{R}^{-1}T_r k \tag{11.139}$$

for large r negative.

An easy computation gives

$$\mathcal{R}^{-1}T_r k = (\delta^{\frac{n-3}{2}}(t-s'+r-x\cdot\theta), \delta^{\frac{n-1}{2}}(t-s'+r-x\cdot\theta) \tag{11.140}$$

and therefore $U_V(-r)\mathcal{R}^{-1}T_r k$ is just the Cauchy data of the solution of $P_V u$ with initial data $(\delta^{\frac{n-3}{2}}(t-s'+r-x\cdot\theta), \delta^{\frac{n-1}{2}}(t-s'+r-x\cdot\theta))$ proving the claim (11.137).

Now we shall prove that

$$MM^*(\delta(t)\otimes f(x))|_{t=0} = B_{n-3}f \tag{11.141}$$

where M^* denotes the formal adjoint of M extended to distributions in the standard fashion, and $\delta(t)\otimes f(x)$ denotes the tensor product of the distributions. Assuming (11.141) for a moment we finish the proof of the Theorem. We have

$$MM^*(\delta(t)\otimes f(x))|_{t=0} = (U_V(t)\mathcal{R}^{-1}W_-^{-1}) \circ (U_V(t)\mathcal{R}^{-1}W_-^{-1})^*(\delta(t)\otimes f). \tag{11.142}$$

Using the unitary of the different operators in the left hand side of (11.142) we obtain

$$MM^*(\delta(t)\otimes f(x)) = U_V(t)U_V^*(t)(\delta(t)\otimes f(x))|_{t=0}. \tag{11.143}$$

Finally an easy calculation left to the reader shows

$$U_V(t)U_V^*(t)(\delta(t)\otimes f)|_{t=0} = f. \tag{11.144}$$

The only remaining loose end in the proof of theorem is then (11.142) which we proceed to prove.

The Schwartz kernel of MM^* is given by

(11.145)
$$K_{MM^*}(t,x,s,x') = \int\limits_{\mathbb{R}} \int\limits_{S^{n-1}} D_s^{\frac{n-3}{2}} u(t-s',x,\omega) D_s^{\frac{n-3}{2}} \overline{u}(-s'+s,x',\omega) d\omega ds.$$

Since both M and M^* are translation invariant we have

(11.146)
$$K_{MM^*}(t,x,s,x') = K_{MM^*}(t-s,x,x').$$

Therefore

(11.147)
$$\widehat{K_{MM^*}}(\lambda,x,x') = \int\limits_{S^{n-1}} \widehat{D_s^{\frac{n-3}{2}}} u(\lambda,x,\omega) d\omega \widehat{D_s^{\frac{n-3}{2}}} \overline{u}(\lambda,x',\omega)$$

where $\wedge$ denotes the Fourier transform in the t-variable. By the definition of the outgoing eigenfunctions we (finally!) get

(11.148)
$$\widehat{K_{MM^*}}(\lambda,x,x') = \int\limits_{S^{n-1}} \lambda^{n-3} \varphi(\lambda,x,\omega) \overline{\varphi}(\lambda,x',d\omega)$$

proving (11.142) and the theorem. ☐

Another way of looking at the amplitude is via Friedlander's radiation field which we proceed to describe

THEOREM 11.14. *Let $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ odd with no bound states, then*

(11.149)
$$\mathcal{K}_V(t,\theta,w) = \lim_{r\to\infty} r^{\frac{n-1}{2}} u(t+r,r\theta,w)$$

with n solution of the continuation problem (1.1).

PROOF. As in (11.8) we write

(11.150)
$$u = \delta(t - x\cdot\omega) + w(t,x,\omega)$$

and

(11.151)
$$w(t,x,w) = 2^{-\frac{1}{2}} (\pi)^{-\frac{(n-1)}{2}} \int\limits_{|x\cdot\theta'|>\rho} k_V(t - x\cdot\theta', \theta', \omega) d\theta'.$$

Therefore

(11.152)
$$w(t+r,r\theta,w) = 2^{-\frac{1}{2}} (\pi)^{-\frac{(n-1)}{2}} \int\limits_{|r\theta\cdot\theta'|>\rho} D_t^{\frac{n-3}{2}} k_V(t+r-r\theta\cdot\theta', \theta', \omega) d\theta'.$$

Of course all of these identities are to be understood in the sense of distributions. As in (2.44) we write

(11.153)
$$\theta' = \alpha\theta + \sqrt{1-\alpha^2}\theta^\perp,$$

where $\theta^\perp$ denotes a direction perpendicular to θ. Then

(11.154)
$$\begin{aligned} w(t+r,r\theta,\omega) = \int\limits_{\mathbb{S}^{n-2}} \int\limits_{|r\alpha|>\delta} & D_t^{\frac{n-3}{2}} k_V\big((t+r)(1-\alpha), \alpha\theta \\ & + \sqrt{1-\alpha^2}\theta^\perp, \omega\big) (1-\alpha^2)^{\frac{n-3}{2}} d\alpha d\theta^\perp. \end{aligned}$$

We can restrict our attention in the integral in (11.154) to those α so that $\tau = r(1-\alpha)$ is sufficiently small for r large since $k_V(s,\theta,\omega)$ is 0 for s sufficiently small and exponentially decaying (see (8.67)) for s sufficiently large. Making the substitution $r(1-\alpha)=\tau$ we obtain for all $\delta>0$ small

$$w(t+r, r\theta, \omega) =$$

$$r\int_{\mathbb{S}^{n-2}}\int_{|T|\leq\delta} D_t^{\frac{n-3}{2}} k_V(t+\tau, (\frac{1-\tau}{r})\theta^{\perp}\sqrt{1-(\frac{1-\tau}{r})^2}\theta^{\perp}, \omega) \tag{11.155}$$

$$(\frac{\tau}{r})^{\frac{n-3}{2}}(1+\frac{1-\tau}{r})^{\frac{n-3}{2}} dT d\theta^{\perp} + O(e^{-\beta r}),\ \beta>0$$

we integrate by parts in the τ-variable $\frac{n-3}{2}$ times and we let $r\to\infty$ proving the claim. (I need to check the constants here too.) □

CHAPTER 12

The scattering amplitude at fixed energy

In Chapter 11 we discussed the high frequency limit of the scattering amplitude by using the propagation of singularities of solutions of $P_V u = 0$ or equivalently. The large frequency behavior of the outgoing eigenfunctions φ as in (11.4).

In order to study the behavior of the scattering amplitude at a fixed energy we use a different class of solutions which are perturbations of growing exponentials of the form $e^{x\cdot\rho}$, $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$.

The approach we will follow will be to check that if the scattering amplitudes of two potential V_1, V_2 at a fixed frequency λ coincide then outside a fixed ball than the outgoing eigenfunctions coincide and therefore its boundary values and the normal derivatives of the outgoing eigenfunctions coincide on the boundary of the ball. Therefore the so called Dirichlet to Neumann map associated to $V_1 - \lambda$, $V_2 - \lambda$ coincide. Then the result follows by combining the (known) fact that the outgoing eigenfunctions are dense on the surface of the sphere and the fact that the Dirichlet to Neumann map λ uniquely determines the potential. We first discuss the Dirichlet to Neumann map. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $q \in L^\infty(\Omega)$ and assume that 0 is not an eigenvalue of $\Delta + q$.

Given $f \in H^{\frac{1}{2}}(\partial\Omega)$ we can solve uniquely the Dirichlet problem

$$(\Delta + q)u = 0 \text{ in } \Omega \tag{12.1}$$

$$u|_{\partial\Omega} = f. \tag{12.2}$$

The Dirichlet to Neumann map is then defined by

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_{\partial\Omega} \tag{12.3}$$

where ν denote the unit outer normal at the boundary. It is easy to check that if q is real-valued, then Λ_q is a self-adjoint map

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega). \tag{12.4}$$

The question that we will address first is whether knowledge of the map Λ_q determines q uniquely.

The difficult part is to find a systematic way of producing boundary values f which would given information about q in Ω. Instead of doing that, Calderón looked at the following quadratic form

$$Q_q(f,g) = \int_\Omega quv + \nabla u \cdot \nabla u \tag{12.5}$$

where u, v are solutions of

$$(\Delta + q)u = (\Delta + q)v = 0 \text{ in } \Omega \tag{12.6}$$

$$u|_{\partial\Omega} = f; v|_{\partial\Omega} = g. \tag{12.7}$$

The point is that Q_q gives the same information as Λ_q since integration by parts shows that

$$Q_q(f,g) = \int_{\partial\Omega} \Lambda_q(f) g dS \tag{12.8}$$

where dS is surface measure on $\partial\Omega$. So instead of trying to find boundary values that will provide such information on q we try to find solutions of $(\Delta + q)u = 0$ that will give information.

Calderón looked at the linearization of the map

$$q \xrightarrow{Q} Q_q \tag{12.9}$$

at $q = 0$.

We have that the Frechet derivative of Q at $q = 0$ is

$$\lim_{\varepsilon\to 0}\left(\frac{Q_{\varepsilon\varphi} - Q_0}{\varepsilon}\right)(f,g) = \int_\Omega \varphi u v \tag{12.10}$$

where u, v are harmonic functions such that $u|_{\partial\Omega} = f$, $v|_{\partial\Omega} = g$.

Now Calderón took

$$u = e^{x\cdot\rho}, \quad v = e^{-x\cdot\overline{\rho}} \tag{12.11}$$

where $\rho \in \mathbb{C}^n$ satisfy

$$\rho \cdot \rho = 0. \tag{12.12}$$

Condition (12.12) implies if $\rho = \eta + ik$ with η, $k \in \mathbb{R}^n$ that $\langle \eta, k\rangle = 0$ and $|\eta| = |k|$ substituting (12.11) in (12.10) we get that if the Frechet derivative of Q at $q = 0$ in the direction φ is 0, then

$$\int_\Omega \varphi e^{2ix\cdot k} = 0 \quad \forall k \in \mathbb{R}^n \tag{12.13}$$

which implies that $\varphi = 0$ in Ω by the Fourier inversion formula.

Motivated by Calderón's approach and the geometrical optics type constructions as in the progressive wave expansion, solutions of $(\Delta + q)u = 0$ were constructed in [ ] that approach complex exponentials $e^{x\cdot\rho}\rho \in \mathbb{C}^n$, $\rho\cdot\rho = 0$ for large $|\rho|$. More precisely:

Theorem 12.15. *Let $q \in L^\infty(\Omega)$, $q = 0$ in Ω^c. Then for every $\rho \in \mathbb{C}^n$ with $\rho\cdot\rho = 0$ and $|\rho| > \|(1+|x|^2)^{\frac{1}{2}} q\|_{L^\infty}$, then there exists solution to*

$$(\Delta + q)u = 0 \text{ in } \mathbb{R}^n \tag{12.14}$$

of the form

$$u(x,\rho) = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{12.15}$$

where

$$\psi_q \to 0 \tag{12.16}$$

uniformly in compact sets for large $|\rho|$.

For a more precise statement and a proof of this result see the Appendix.
We shall use the solutions (12.15) to prove

THEOREM 12.16. *Let* $q_1, q_2 \in L^\infty(\Omega), \partial\Omega\ C^\infty, \subseteq \mathbb{R}^n, n \geq 3$ *such that* 0 *is not an eigenvalue for* $\Delta \neq q_1$ *or* $\Delta + q_2$. *Assume*

$$\Lambda_{q_1} = \Lambda_{q_2}. \tag{12.17}$$

Then

$$q_1 = q_2 \text{ in } \Omega. \tag{12.18}$$

PROOF. The following identity follows immediately using Green's theorem

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2})(f_1) f_2 \tag{12.19}$$

where u_i, for $i = 1, 2$ satisfy

$$(\Delta + q_i) u_i = 0 \text{ in } \Omega \tag{12.20}$$

$$u_i|_{\partial\Omega} = f_i. \tag{12.21}$$

If $\Lambda_{q_1} = \Lambda_{q_2}$, then

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0 \tag{12.22}$$

for every solution $u_i \in H^1(\Omega)$ of $(\Delta + q_i)u_i = 0$, $i = 1, 2$. We take

$$\begin{aligned} u_1 &= e^{x\cdot\rho_1}(1 + \psi_{q_1}(x, \rho_1)) \\ u_2 &= e^{x\cdot\rho_2}(1 + \psi_{q_2}(x, \rho_2)) \end{aligned} \tag{12.23}$$

as in (12.15) with u_i solution of $(\Delta + q_i)u_i = 0$ and

$$\rho_1 = \eta + i(rw + k), \ \ \rho_2 = -\eta - i(rw + k) \tag{12.24}$$

with η, w, $k \in \mathbb{R}^n$, $r \in \mathbb{R}$ with $\langle \eta, w\rangle = \langle \eta, k\rangle = \langle w, k\rangle = 0$ and $|\eta|^2 = |w|^2 + |k|^2$. Substituting (12.23) with ρ_i as in (12.24) we obtain

$$\int_\Omega (q_1 - q_2) e^{2ix\cdot k}(1 + \psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2}) = 0 \tag{12.25}$$

Letting $r \to \infty$ and using (12.16) we get

$$\int_\Omega (q_1 - q_2) e^{2ix\cdot k} = 0 \tag{12.26}$$

which implies $q_1 = q_2$ in Ω, proving the theorem. □

Now we relate the Dirichlet to Neumann map for $V - \lambda$ to the scattering amplitude $a_V(\lambda, \theta, \omega)$. The main result is

THEOREM 12.17. *Let* $V_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, 2$ *such that*

$$a_{V_1}(\lambda, \theta, w) = a_{V_2}(\lambda, \theta, w) \tag{12.27}$$

for all $\theta, \omega, \in C^{n-1}$ *and a fixed* $\lambda \in \mathbb{R} - 0$. *Then* $V_1 = V_2$.

PROOF. In the proof we shall use the following lemma.

LEMMA 12.39. *Let* $\lambda \in \mathbb{R} - 0$ *such that*

$$(\Delta - \lambda)u = f, \quad f \in C_0^\infty(\mathbb{R}^n) \tag{12.28}$$

and

$$u = 0(|x|^{-\frac{(n-1)}{2}-1}), \tag{12.29}$$

then

$$u \in C_0^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} u \subseteq \operatorname{supp} f. \tag{12.30}$$

PROOF. This is a form of Rellich's lemma. First of all by elliptic regularity we know that $u \in C^\infty(\mathbb{R}^n)$ and moreover by (12.29), $u \in L^2(\mathbb{R}^n)$. Taking Fourier transform of both sides of (12.28) we get that

$$(|\xi|^2 - \lambda)\hat{u} = \hat{f}. \tag{12.31}$$

If $\lambda < 0$, then

$$\hat{u} = \frac{\hat{f}}{|\xi|^2 - \lambda} \tag{12.32}$$

By using the Paley-Wiener theorem since $f \in C_0^\infty(\mathbb{R}^n)$, we obtain $u \in C_0^\infty(\mathbb{R}^n)$. If $\lambda > 0$, then

$$\hat{u} = \frac{\hat{f}}{|\xi|^2 - \lambda}, \quad |\xi|^2 \neq \lambda \tag{12.33}$$

If $\hat{f} = 0$ on $|\xi|^2 - \lambda = 0$, then since $\hat{f}$ has an analytic extension, this would imply that $\hat{u}$ is analytic. Again using Paley-Wiener we get $u \in C_0^\infty(\mathbb{R}^n)$. If $\hat{f}(\xi_0) \neq 0$ for some $\xi_0 \neq 0$, then it is easy to deduce from (12.33) that

$$\hat{u} \notin L^2(\mathbb{R}^n). \tag{12.34}$$

□

LEMMA 12.40. *Let* $\varphi(\lambda, x, \omega)$ *be the outgoing eigenfunctions for* $V \in C_0^\infty(\mathbb{R}^n)$ *as in* (11.4). *Then* $\{\varphi(\lambda, x, \omega)\}_{x \in \partial B(0,\rho)}$ *is dense in* $L^2(\partial B(0, R))$ *if* $\operatorname{supp} V \subseteq B(0, \rho)$.

PROOF. I will only sketch it here. I hope to find one where I don't have to prove so much. Assume without loss of generality that $\partial B(0, \rho) = \partial B(0, 1) = S^{n-1}$. Assume $\exists f \in L^2(S^{n-1})$ such that

$$\int_{S^{n-1}} f(\theta)\varphi(\lambda, \theta, \omega)d\theta = 0, \forall \omega \in S^{n-1} \tag{12.35}$$

and fixed $\lambda \in \mathbb{R} - 0$.

Let $G(\lambda, x, y)$ be the Green's kernel of $\Delta + V$ satisfying the Sommerfeld radiation condition

$$(\Delta + V - \lambda)G = \delta(x - y) \tag{12.36}$$

$$r^{\frac{n-1}{2}} \frac{d}{dr} G - i\lambda G = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right) \text{ as } |x| \to \infty \tag{12.37}$$

(with $r = |x|$).

This can be proven in a similar fashion to the expansion (11.4) using the wave equation (more details here).

We define as a singular integral.

$$w(x) = \int_{S^{n-1}} G(x, \theta, \lambda) f(\theta) d\theta \tag{12.38}$$

with G as in (12.36) and (12.37) and f as in (12.35). Then we can check (the proof is as in the expansion (11.4) for the outgoing eigenfunction φ) that

$$G(x, y, \lambda) = \frac{i|\lambda||x|}{|x|^{\frac{n-1}{2}}} \varphi(\lambda, y, \frac{x}{|x|}) + 0(|x|^{-\frac{(n-1)}{2}-1}). \tag{12.39}$$

Therefore as $|x| \to \infty$ by (12.35)

$$w(x) = 0(|x|^{-\frac{(n-1)}{2}-1}) \tag{12.40}$$

Now w satisfies

$$(\Delta - \lambda) w = -V_w \in C_0^\infty(\mathbb{R}^{n-1}). \tag{12.41}$$

Therefore by Lemma 12.39 $w = 0$ in $\complement B(0,1)$.

Now we can assume that λ is not an eigenvalue of $\Delta + V$ in $B(0,1)$ (this just requires to enlarge a bit the ball we are working on)

Since w satisfies (12.41) and λ is not an eigenvalue of $\Delta + V$ we get that

$$w = 0 \text{ in } \Omega \tag{12.42}$$

Then we get that $w = 0$. However the jump of w across $B(0,1)$ is given by

$$[w] = f = 0, \tag{12.43}$$

finishing the proof of the Lemma. □

Now we are in position to end the proof of Theorem (12.17).

Assume

$$a_{V_1}(\lambda, \theta, \omega) = a_{V_2}(\lambda, \theta, \omega),\ \forall\ \theta, \omega \in \mathbb{S}^{n-1},\ \lambda \in \mathbb{R} - 0 \text{ fixed.} \tag{12.44}$$

Then by (11.4) we have that the corresponding generalized eigenfunctions to V_1, V_2 satisfy

$$\varphi_{V_1} - \varphi_{V_2} = 0(|x|^{-\frac{(n-1)}{2}-1}) \tag{12.45}$$

Now $\varphi_{V_1} - \varphi_{V_2}$ satisfies

$$(\Delta - \lambda)(\varphi_{V_1} - \varphi_{V_2} = -V_1 \varphi_{V_1} + V_2 \varphi_{V_2} \tag{12.46}$$

Using Lemma 12.39 we have that

$$\varphi_{V_1} - \varphi_{V_2} \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ \varphi_{V_1} - \varphi_{V_2} = 0 \text{ for } |x| \geq R. \tag{12.47}$$

Therefore we have that

$$\Lambda_{V_1 - \lambda}(\varphi_{V_1}) = \Lambda_{V_2 - \lambda}(\varphi_{V_2}) \tag{12.48}$$

where the Dirichlet to Neumann map is defined in $B(0,R)$ and $V_1 - \lambda$, $V_2 - \lambda$ are extended to be zero outside the support of V_1 and support of V_2 respectively.

Now by Lemma 12.40 we conclude that

$$\Lambda_{V_1 - \lambda}(f) = \Lambda_{V_2 - \lambda}(f)\ \forall f \in C^\infty(B(0,R)) \tag{12.49}$$

and now using Theorem (12.16) we conclude that

$$V_1 = V_2 \tag{12.50}$$

proving the Theorem. □

Appendix – Complex geometrical optics

In this appendix we prove Theorem (12.15). The proof is as in [ ].

We first introduce weighted spaces that will be useful in showing also uniqueness of the solutions of the equation (11.1) since one is imposing decay conditions at infinity.

DEFINITION 12.11. $L^2_\delta(\mathbb{R}^n)$ is the Banach space with norm

$$\|f\|_{L^2_\delta} = \int_{\mathbb{R}^n} (1+|x|^2)^\delta |f(x)|^2 dx \tag{12.51}$$

and the corresponding weighted Sobolev space is defined by $W^s_\delta(\mathbb{R}^n)$ a Banach space with norm

$$\|f\|_{w^s_s} = \|(1+|x|^2)^{\frac{\delta}{2}} f\|_{H^s(\mathbb{R}^n)}. \tag{12.52}$$

THEOREM 12.18. *Suppose $q \in L^\infty(\mathbb{R}^n)$ has compact support and let $\rho\varepsilon\mathbb{C}^n$ with $\rho \cdot \rho = 0$. Let $-1 < \delta < 0$. Then there exists $\varepsilon > 0$ such that if*

$$\frac{\|(1+|x|^2)^{\frac{1}{2}} q\|}{|\rho|}_{L^\infty} < \varepsilon, \tag{12.53}$$

then there exists a unique solution u to

$$(\Delta + q)u = 0 \qquad \text{in } \mathbb{R}^n \tag{12.54}$$

of the form

$$u = e^{x\cdot\rho}\big(1 + \psi_q(x,\rho)\big) \tag{12.55}$$

with $\psi_q \varepsilon L^2_\delta(\mathbb{R}^n)$. Furthermore

$$\|\psi_q\|_{W^s_\delta} \le \frac{C(\zeta,\varepsilon)}{|\rho|}\|q\|_{W^s_{\delta+1}} \qquad s \ge 0. \tag{12.56}$$

The proof will be shown to be an easy consequence of the following.

PROPOSITION 12.34. *Let $f\varepsilon L^2_{\delta+1}$, $-1 < \delta < 0$. Then there exists a unique solution $\psi\varepsilon L^2_\delta(\mathbb{R}^n)$ to*

$$(\Delta + 2\rho\cdot\nabla)\psi = f \tag{12.57}$$

with

$$\|\psi\|_{W^s_\delta} \le \frac{\subset(\zeta)}{|\rho|}\|f\|_{W^s_\delta} \qquad s \ge 0. \tag{12.58}$$

PROOF. Proof of Theorem A.3 using Proposition 8.20 (i)**Existence.** This is just a Neumann series type argument. Let $\psi_{-1} = 1$.

$$(\Delta + 2\rho\cdot\nabla)\psi_\delta = q\psi_{j-1} \qquad j \ge 1. \tag{12.59}$$

We can solve (12.59) since $q\psi_{\delta-1}$ has compact support by Proposition 8.20 and moreover

$$\|\psi_\delta\|_{L^2_J} \le \frac{C}{|\rho|}\|q\psi_{j-1}\|_{L^2_{J+1}} \qquad j \ge 0. \tag{12.60}$$

Now $q\psi_{j-1} = q(1+|x|^2)^{\frac{1}{2}}(1+|x|^2)^{\frac{-1}{2}}\psi_{j-1}$ and by definition (12.51), $(1+|x|^2)^{\frac{-1}{2}}\psi_{j-1}\varepsilon L^2_\delta$. Therefore from (12.60) we conclude

$$\|\psi_j\|_{L^2_\delta} \leq \frac{C}{|\rho|}\|q(1+|x|^2)^{\frac{1}{2}}\|_{L^\infty}\|\psi_{j-1}\|_{L^2_\delta} \tag{12.61}$$

and therefore by induction

$$\|\psi_j\|_{L^2_\delta} \leq \Big(\frac{C\|q(1+|x|^2)^{\frac{1}{2}}\|_{L^\infty}}{|\rho|}\Big)^j C\frac{\|q\|}{|\rho|}L^2_\delta. \tag{12.62}$$

Now we take

$$\psi = \sum_{j=1}^{\infty}\psi_j. \tag{12.63}$$

We have

$$\|\psi\|_{L^2_\delta} \leq C\frac{\|q\|_{L^2_\delta}}{|\rho|}\sum_{j=1}^{\infty}\Big(\frac{C\|q(1+|x|^2)^{\frac{1}{2}}\|_{L^\infty}}{|\rho|}\Big)^j \tag{12.64}$$

The series (12.22) then converges geometrically if we choose $\varepsilon < \frac{1}{C}$.

The estimate for higher Sobolev weighted spaces is obtained in a similar fashion.

(ii)**Uniqueness.**

Let ψ_1, $\psi_2\varepsilon W^S_S(\mathbb{R}^n)$ be solutions of (12.57) satisfying (12.58). Then

$$\Delta(\psi_1-\psi_2)+\rho\cdot\nabla(\psi_1-\psi_2) = q(\psi_1-\psi_2) \tag{12.65}$$

and

$$\|\psi_1-\psi_2\|_{W^s_\delta} \leq C\frac{\|1+|x|^2)^{\frac{1}{2}}q\|_{L^\infty}}{|\rho|}\|\psi_1-\psi_2\|_{W^s_\delta}, s\geq 0. \tag{12.66}$$

Then by the choice of (ρ)

$$\|\psi_1-\psi_2\|_{W^s_\delta} < \|\psi_1-\psi_2\|_{W^s_\delta}, \qquad s\geq 0 \tag{12.67}$$

concluding that $\psi_1=\psi_2$.

Proof of Proposition 8.20

(i)**Uniqueness.**

Suppose $w\varepsilon L^2_\delta(\mathbb{R}^n), -1<\delta<0$ with

$$\Delta w + 2\rho\cdot\nabla w = 0. \tag{12.68}$$

Then since $w\in S'(\mathbb{R}^n)$ by taking Fourier transform we conclude

$$(-|\xi|^2+2i\rho\cdot\xi)\widehat{w}(\xi) = 0. \tag{12.69}$$

Therefore we have

$$\operatorname{supp}\widehat{w}\subset\{\xi; |\xi|^2+2i\rho\cdot\xi=0\}=\mathcal{M}. \tag{12.70}$$

It is easy to see that $\mathcal{M}$ is a codimension 2 sphere. Now the result will follow from the following lemma which appears in Hörmander (Vol. I).

Lemma 12.41. *Let $u\in S'\cap L^2_{\text{loc}}$ and suppose that*

$$\limsup_{R\to\infty}\frac{1}{R^k}\int\limits_{|x|<R}|u|^2dx<\infty. \tag{12.71}$$

If in addition, $\widehat{u}$ is supported on a C^1 submanifold $\mathcal{M}$ of codimension k, then $\widehat{u}$ is an L^2 density $\widehat{u_0}dS$ on $\mathcal{M}$ where dS denotes surface measure and $\exists C > 0$, such that

$$\int\limits_M |\widehat{u_0}|^2 dS \leq C \limsup_{R\to\infty} \frac{1}{R^k} \int\limits_{|x|<R} |u|^2 dx. \tag{12.72}$$

Now for $w \in L^2_\delta$ satisfying (12.69) we have

$$\|u^2\|_{L^2_\delta} \geq \int\limits_{|x|<R} (1+|x|^2)^\delta |u(x)|^2 dx, \tag{12.73}$$

Therefore

$$\|u\|^2_{L^2_\delta} \geq R^{2\delta} \int\limits_{|x|<R} |u(x)|^2 dx \tag{12.74}$$

and finally

$$R^{-2-2\delta}\|u\|^2_\delta \geq R^{-2} \int\limits_{|x|<R} |u(x)|^2 dx. \tag{12.75}$$

From (12.75)

$$\limsup \frac{1}{R^2} \int\limits_{|x|<R} |u(x)|^2 dx = 0 \tag{12.76}$$

and by (12.72) we deduce finally

$$\widehat{u_0} = 0 \text{ and therefore } u = 0. \tag{12.77}$$

(ii)**Existence.** To do this we shall make a microlocal partition of the submanifold $\mathcal{M}$. We can choose

$$\rho = s(e_1 + ie_2) \tag{12.78}$$

where $e_1 = (1, 0, \cdots 0), e_2 = (0, 1, 0 \cdots 0)$ and $s > 0$. Let

$$l(\xi) = -|\xi|^2 + 2i\rho\cdot\xi = \xi_1^2 + (\xi_2 - s)^2 + \cdots + \xi_n^2 - s^2 + 2is\xi_1 \tag{12.79}$$

Let $N_s(\mathcal{M})$ be the s-tubular neighborhood of $\mathcal{M}$ with $\mathcal{M}$ as in (12.70). We now define an open covering of $\mathbb{R}^n$ by

$$U_1 = \mathbb{R}^n - N_{\frac{s}{2}}(\mathcal{M}) \tag{12.80}$$

$$U_2 = \{\xi \varepsilon \mathbb{R}^n; |\xi_2 - s| \geq \frac{s}{2}\} \cap N_s(\mathcal{M}) \tag{12.81}$$

and U_j for $j \neq 1, 2$ by

$$U_j = \{\xi \varepsilon \mathbb{R}^n, |\xi_j| \geq \frac{s}{2}\} \cap N_s(M). \tag{12.82}$$

Let $\chi_j(\xi)$ be a partition of unity subordinate to the U_j. We set

$$\widehat{w_j}(\xi) = \frac{\chi_j(\xi)\widehat{w}(\xi)}{l(\xi)}, w = \Sigma w_j. \tag{12.83}$$

On U_1 we have $|l(\xi) \geq sC$ for $C > 0$. Therefore

$$\|w_1\|_{L^2_\delta} \leq \|w_1\|_{L^2} \leq \frac{1}{sC}\|f\|_{L^2} \leq \frac{1}{sC}\|f\|_{L^2_{\delta+1}}. \tag{12.84}$$

Now on U_j, $j \geq 1$, we define new coordinates by

$$eta_1 = \xi_1,\ \eta_k = \xi_k,\ k \neq j\eta_j = \frac{\xi_1^2 + (\xi_2 - s)^2 + \cdots + \xi_n^2 - s^2}{s}. \tag{12.85}$$

Of course in these new coordinates

$$l(\eta) = s(\eta_2 + i\eta_1). \tag{12.86}$$

Clearly $\frac{\partial \eta}{\partial \xi}$ and $\left(\frac{\partial \eta}{\partial \xi}\right)^{-1}$ are bounded on U_j, $j > 1$. We shall use the following result which is in [ ] and it is the invariance of the weighted spaces under change of variables (12.46).

LEMMA 12.42. *Assume η is an invertible map $\eta : U_j \to V_j$, where $V_j \leq \mathbb{R}^n$ is open also $\left(\frac{\partial \eta}{\partial \xi}\right)$, $\left(\frac{\partial \eta}{\partial \xi}\right)^{-1}$ are uniformly bounded by a constant M. Then $\exists C(M) > 0$ such that if* $\operatorname{supp} \widehat{f} \subset U^j$ *and* $\operatorname{supp} \widehat{g} \subset V^j$ *then for all δ, $-1 \leq \delta \leq 1$,*

$$\begin{aligned} \|(\widehat{f} \circ \eta)^\vee\|_{L^2_\delta} &\leq C(M)\|f\|_{L^2_\delta} \\ \|(\widehat{g} \circ \eta^{-1})^\vee\|_{L^2_\delta} &\leq C(M)\|g\|_{L^2_\delta}. \end{aligned} \tag{12.87}$$

Sketch of proof.

For $\delta = 1$, $\|f\|_{L^2_1} = \|\widehat{f}\|_{H^1}$ and then the computation follows by the chain rule.

For $\delta = -1$ one uses duality and the $\delta = 1$ case (see [N-W] for more details). The remaining argument $< \delta < 1$ follows by interpolation. ☐

Now the estimate (12.56) just follows from

LEMMA 12.43. *The map*

$$\widehat{f}(\xi) \xrightarrow{Z_l} \frac{\widehat{f}(\xi)}{\xi_l + i\xi_1} \tag{12.88}$$

is bounded from $L^2_{\delta+1}(\mathbb{R}^n) \to L^2_\delta(\mathbb{R}^n)$ for $-1 < \delta < 0$.

PROOF. The result can be seen estimating directly since Z_l is just the solution operator for the Cauchy Riemann equations. We know $\left(\frac{1}{\xi_l + i\xi_j}\right)^\vee = C\frac{1}{x_l + ix_j}$ for some constant C and now one estimates the convolution with $\frac{1}{x_l + ix_1}$ using Cauchy Schwartz.

A more elegant alternative is first we assume without loss of generality that $\widehat{f}(0) = 0$. We write (take $l = 2$)

$$\frac{\widehat{f}(\xi)}{\xi_2 + i\xi_1} = \frac{f_1(\xi)\xi_1}{\xi_2 + i\xi_1} + \frac{f_2(\xi)\xi_2}{\xi_2 + i\xi_1} \tag{12.89}$$

where

$$f_1(\xi) = \int_0^1 \frac{\partial f}{\partial \xi_1}(t\xi_1, \xi_2)dt \tag{12.90}$$

and

$$f_2(\xi) = \int_0^1 \frac{\partial f}{\partial 2\xi_2}(\xi_1, s\xi_2)ds. \tag{12.91}$$

Now

$$\left\| \frac{\widehat{f}(\xi)}{\xi_2 + i\xi_2} \right\|_{L^2} \leq C(\|f_1\|_{L^2} + \|f_2\|_{L^2}\|) \leq C\|\widehat{f}\|_{H^1}. \tag{12.92}$$

So

$$\|\psi\|_{L^2} \leq C\|f\|_{L^2_1} \tag{12.93}$$

that is Z_l is bounded for $L^2_1 \to L^2_0$. By duality Z_l is bounded from $L^2_0 \to L^2_{-1}$ and then by interpolation Z_l is bounded from $L^2_{\delta+1} \to L^2_\delta$. (This duality arguments and interpolation arguments are in Nirenberg-Walker, maybe one should put more details here.) □

CHAPTER 13

Backscattering

We have already shown, in Corollary 10.87, that the scattering transform (for $n \geq 3$, odd)

$$\mathcal{C}_c^\infty(\mathbb{R}^n) \ni V \longmapsto a_V \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times [\mathbb{C}\backslash\{\lambda_j\}]) \tag{13.1}$$

is injective, i.e. a_V determines V. This is not unduly surprising since, ignoring the meromorphy in λ, a_V is a function of $(n-1)+(n-1)+1 = 2n-1$ variables, from which we seek to recover the potential, a function of n variables.

A more challenging, formally determined, problem is to consider instead the *backscattering transform*

$$B : \mathcal{C}_c^\infty(\mathbb{R}^n) \ni V \longmapsto a_V(-\theta, \theta; \lambda) \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \times [\mathbb{C}\backslash\{\lambda_j\}]). \tag{13.2}$$

The question then arises as to whether (13.2) is injective. Ideally one would like to determine precisely the range of B and give a recovery procedure. We will not manage to do this, but we shall show that B is, near almost every V, locally invertible.

To do so it is very convenient to replace the space $\mathcal{C}_c^\infty(\mathbb{R}^n)$, with its rather intricate topology, by a Hilbert space. This we do by allowing V to become more singular, after first fixing the size of the support. For $\rho \in (0, \infty)$ consider

$$\begin{aligned} \dot{H}^{\frac{n+1}{2}}(B(\rho)) = \{V \in L^2(\mathbb{R}^n); & V(x) = 0 \text{ in } |x| > \rho, \\ & D^\alpha V \in L^2 \quad \forall\ |\alpha| \leq \frac{n+1}{2}\}. \end{aligned} \tag{13.3}$$

As usual we are restricting ourselves to n odd.

The choice of Sobolev order here is not very critical - it is convenient that $\frac{n+1}{2}$ is an integer and rather more important that $\frac{n+1}{2} > \frac{n}{2}$. The latter condition means that $\dot{H}^{\frac{n+1}{2}}B(\rho))$ is an *algebra*.

Lemma 13.44. *(Gagliardo-Nirenberg, see* [**4**]*) For any $k \in \mathbb{N}$ with $k > n/2$ and any $s \in \mathbb{R}$ satisfying $-k \leq s \leq k$*

$$H^k(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) \subset H^s(\mathbb{R}^n). \tag{13.4}$$

In particular if $s \in \mathbb{R}$ and $-\frac{n+1}{2} \leq s \leq \frac{n+1}{2}$ then

$$\dot{H}^{\frac{n+1}{2}}(B(\rho)) \cdot H^s(\mathbb{R}^n) \subset \dot{H}^s(B(\rho)). \tag{13.5}$$

The main point of this lemma is that we can then have V acting as a multiplication operator on these Sobolev spaces.

LEMMA 13.45. *For any k ($\in \mathbb{Z}$ for simplicity) the normalized Radon transform gives a bounded map*

$$R_n : \dot{H}^k(B(\rho)) \longrightarrow \dot{H}^k([-\rho,\rho]\times\mathbb{S}^{n-1}) = \{u \in H^k(\mathbb{R}\times\mathbb{S}^{n-1});\ u(s,\theta)=0 \text{ in } |s|>\rho\}. \tag{13.6}$$

PROOF. This was shown, for $k=0$, in Lemma 3.10 as a consequence of the L^2 boundedness of the Fourier transform. Consider the case $k>0$. We know that R (and hence R_n) intertwines Δ with D_s^2. Thus if $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then

$$D_s^2 R_n f = R_n \Delta f \tag{13.7}$$

Since we actually know that R_n is a partial isometry on L^2,

$$\langle R_n f, D_s^2 R_n f\rangle_{L^2} = \langle \Delta f, f\rangle. \tag{13.8}$$

By continuity then, $f \in \dot{H}^1(B(\rho)) \Longrightarrow D_s R_n f \in L^2$. Repeating this argument a finite number of times shows that

$$f \in \dot{H}^k(B(\rho)) \Longrightarrow D_s^j R_n f \in L^2([-\rho,\rho]\times\mathbb{S}^{n-1}) \quad 0 \le j \le k. \tag{13.9}$$

To get tangential regularity, suppose that W is a $\mathcal{C}^\infty$ vector field on the sphere. Then

$$\begin{aligned} WR_nf(s,\theta) &= c_n D_s^{\frac{n-1}{2}} W \int \delta(s-x\cdot\theta) f(x)dx \\ &= \sum_{j=1}^n q_j(\theta) D_s R_n(x_j f), \quad W(x\cdot\theta) = \sum_{j=1}^n x_j q_j(\theta). \end{aligned} \tag{13.10}$$

Thus $WR_nf \in L^2$. Repeating this argument we conclude that (13.6) holds for $k \ge 0$.

The same type of argument applies to R_n^t. Thus

$$R_n^t u(x) = c_n \int_{\mathbb{S}^{n-1}} \delta(s-x\cdot\omega) D_s^{\frac{n-1}{2}} u(s,\omega) ds \tag{13.11}$$

is bounded from $L^2([-\rho,\rho]\times\mathbb{S}^{n-1})$ into $L^2(B(\rho))$. Direct differentiation therefore shows that it is bounded from $H^k([-\rho,\rho]\times\mathbb{S}^{n-1})$ into $H^k(B(\rho))$ for $k \in \mathbb{N}$. By duality it therefore follows that (13.6) holds for $k \in -\mathbb{N}$, and hence for all $k \in \mathbb{Z}$ as claimed. □

EXERCISE 13.7. From the proof above,

$$\begin{aligned} R^t : \{u \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}); D_s^j u \in L^2(\mathbb{R}\times\mathbb{S}^{n-1})\} \\ \longrightarrow H^k(B(\rho)) \text{ if } k \ge 0, \text{ and} \end{aligned} \tag{13.12}$$

$$\begin{aligned} R^t : \{u \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{S}^{n-1}); u \in D_s^{-k} L^2(\mathbb{R}\times\mathbb{S}^{n-1})\} \\ \longrightarrow H^k(B(\rho)) \text{ if } k \le 0. \end{aligned} \tag{13.13}$$

That is, one does not need tangential regularity to ensure the regularity of $R_n^t f$.

COROLLARY 13.5. *For any $k \in \mathbb{Z}$ satisfying $\frac{n-1}{2} \ge k \ge -\frac{n+3}{2}$, and any potential $V \in \dot{H}^{\frac{n+1}{2}}(B(\rho))$, V_{LP} gives a bounded map*

$$V_{\mathrm{LP}} : H^k(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow H^{k+1}(\mathbb{R}\times\mathbb{S}^{n-1}). \tag{13.14}$$

PROOF. Recall that $V_{\mathrm{LP}} = c_n^2 D_s^{\frac{n-1}{2}} R \cdot V \cdot R^t D_s^{\frac{n-3}{2}}$. From Remark 13.7,

$$R^t D_s^{\frac{n-3}{2}} : H^k(\mathbb{R} \times \mathbb{S}^{n-1}) \longrightarrow H^{k+1}(B(\rho)). \tag{13.15}$$

Then, from Lemma 13.44, multiplication by V maps into $\dot{H}^{k+1}(B(\rho))$ and from Lemma 13.45, $D_s^{\frac{n-1}{2}} R$ maps into $\dot{H}^{k+1}([-\rho, \rho] \times \mathbb{S}^{n-1})$. □

We shall apply these regularity estimates to show that the backscattering transform extends by continuity to $\dot{H}^{\frac{n+1}{2}}(B(\rho))$. Before doing this we introduce the 'modified backscattering transform,' in which 'excess' information has been discarded.

For $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we know that the scattering kernel (of which a_V is the Fourier transform), κ_V, has support in $\{s \geq -2\rho\}$. Consider the combined restriction, differentiation and projection map

(13.16)

$$\chi_\rho : \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^{n-1}) \overset{D_s^{\frac{n-3}{2}}}{\to} \longrightarrow \mathcal{C}^\infty([-2\rho, 2\rho] \times \mathbb{S}^{n-1})$$
$$\overset{\pi_\rho}{\to}\longrightarrow \overline{D_s^{\frac{n-3}{2}} R_n(\dot{H}^{\frac{n+1}{2}}(B(2\rho))} \subset \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}).$$

Here π_ρ is orthogonal projection, in $H^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1})$, onto the closure of the range of $D_s^{\frac{n-3}{2}}$ following the normalized Radon transform.

Now, for $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we know that

$$\operatorname{singsupp} \kappa_V \subseteq \{s = 0, \theta = \omega\}. \tag{13.17}$$

Thus the backscattering kernel, $\kappa_V(s, -\theta, \theta) \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$. We can therefore apply (13.16) to define the *modified backscattering transform*

$$\beta : \dot{\mathcal{C}^\infty}(B(\rho)) \ni V \longmapsto \chi_\rho[\kappa_V(s, -\theta, \theta)] \in \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}). \tag{13.18}$$

THEOREM 13.19. *The modified backscattering transform* (13.18) *extends, by continuity, to a continuous operator*

$$\beta : \dot{H}^{\frac{n+1}{2}}(B(\rho)) \longrightarrow \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}) \tag{13.19}$$

which is entire analytic, i.e. can be written

$$\beta(V) = \sum_{j=1}^{\infty} \beta_j(V, \dots V) \tag{13.20}$$

where

$$\beta_1 : \dot{H}^{\frac{n+1}{2}}(B(\rho)) \longrightarrow \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}) \tag{13.21}$$

is a linear isomorphism and for each $j \geq 2$

$$\beta_j : [\dot{H}^{\frac{n+1}{2}}(B(\rho))]^j \longrightarrow \dot{H}^{\frac{5}{2}}([-2\rho, 2\rho] \times \mathbb{S}^{n-1}) \tag{13.22}$$

is symmetric and satisfies, for each $0 \leq \epsilon \leq \frac{1}{2}$,

$$\|\beta_j(V, \dots, V)\|_{\frac{5}{2}-\epsilon} \leq \frac{C^{j+1}\|V\|^j}{(j!)^{2\epsilon}}. \tag{13.23}$$

As we shall describe below, this proves that β is almost everywhere a local isomorphism, and hence that B is almost everywhere locally invertible.

EXERCISE 13.8. Is β ($=\beta_\rho$) a global isomorphism? Is the differential of β, at any $V \in \dot{H}^{\frac{n+1}{2}}(B(\rho))$ always invertible (it *is* always Fredholm)? If not, what are the singular points? (Characterize them and show there are none?)

The Taylor expansion (13.20) for the modified backscattering transform is closely related to the Born approximation. This in turn is just the Neumann (or perhaps one should say Volterra) series for the solution of the (Radon-transformed) wave equation. Recall equation (10.43) for the 'correction term' α'. Formally at least we can write this as a series

$$\begin{aligned} \alpha' &= \sum_{j=1}^{\infty} \alpha'_j,\ \alpha'_j = [(D_t + D_s)^{-1} V_{\mathrm{LP}}]^j \alpha'_0 \\ \alpha'_0 &= (D_t + D_s)^{-1} V_{\mathrm{LP}} \delta(t-s)\delta_\theta(\omega). \end{aligned} \tag{13.24}$$

Here $(D_t + D_s)^{-1}$ is the inverse of the forcing problem

$$(D_t + D_s)u = f,\ f = 0 \text{ in } s < -\rho,\ u = 0 \text{ in } s < -\rho \Longrightarrow u = (D_t + D_s)^{-1} f. \tag{13.25}$$

We proceed to show that, for any $V \in \dot{H}^{\frac{n+1}{2}}(B(\rho))$, the series (13.24) converges.

PROPOSITION 13.35. *For any $V \in \dot{H}^{\frac{n+1}{2}}(B(\rho))$, $T < \infty$ and $k \in \mathbb{Z}$ with $-\frac{n+3}{2} \le k \le \frac{n+1}{2}$, as an operator on*

$$\dot{H}^k_{T,\rho} = \left\{ f \in \dot{H}^k([-\infty, T]_t \times [-\rho, \rho]_s \times \mathbb{S}^{n-1}); f = 0 \text{ in } t < -\rho \right\} \tag{13.26}$$

$(D_t + D_s)^{-1} V_{\mathrm{LP}}$ is bounded and for some $C = C(T)$

$$\| [(D_t + D_s)^{-1} V_{\mathrm{LP}}]^j \|_{H^k} \le \frac{C^{j+1} \|V\|^j}{j!} \tag{13.27}$$

where $\|V\|$ is the norm in $\dot{H}^{\frac{n+1}{2}}(B(\rho))$.

PROOF. Since t is a parameter in the action of V_{LP} and $(D_t + D_s)^{-1}$ is bounded on any Sobolev space the boundedness is clear from Corollary 13.5. Only the Volterra-type estimate (13.27) needs to be shown. To carry out this estimation it is convenient to introduce $D_t + D_s$ and D_s as coordinate vector fields, i.e. change coordinates to

$$t' = t, s' = s - t. \tag{13.28}$$

The operators are transformed as follows

$$D_t + D_s \longmapsto D_{t'},\ V_{\mathrm{LP}} \longmapsto V'_{\mathrm{LP}}(t', s', D_{s'}) \tag{13.29}$$

where V'_{LP} is still a non-local operator in s', but now depending on t' as a parameter, i.e.

$$V'_{\mathrm{LP}} u(t', s') \text{ depends only on } u(t', \cdot). \tag{13.30}$$

The iterated operator is therefore

$$\left(D_{t'}^{-1} V'_{\mathrm{LP}} \right)^j. \tag{13.31}$$

from -250 to 250, y from -200 to 250 .5pt ¡4pt¿ [.2,.67] from -200 0 to 200 0 ¡4pt¿ [.2,.67] from 0 -180 to 0 180 -150 -150 150 150 / from 4

FIGURE 1. Plane wave initial data

Applying this $|k|+1$ times to H^k gives a bounded map into the space $C^0([-\rho,T];H^k(\mathbb{S}^{n-1}\times\mathbb{R}_{s'}))$. Then, integration in t' and continuity of V'_{LP} shows that

$$\|(D_{t'}^{-1}V'_{\mathrm{LP}})^{j+|k|+1}u\|_{H^k(\mathbb{S}^{n-1}\times\mathbb{R}_{s'})}(t')\le\frac{C(t'+\rho)^j}{j!}. \tag{13.32}$$

This gives (13.27). □

Of course from Corollary 13.5 we know that, if $-\frac{n+3}{2}\le k\le\frac{n-1}{2}$,

$$(D_t+D_s)^{-1}L_{\mathrm{LP}}:\dot H^k_{T,\rho}\longrightarrow\dot H^{k+1}_{T,\rho} \tag{13.33}$$

Since

$$\delta(t-s)\delta_\theta(\omega)\in H^{-\frac{n+1}{2}}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}) \tag{13.34}$$

it follows that

$$\alpha'_j\in H^{-\frac{n+1}{2}+\min(j,n+1)}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}). \tag{13.35}$$

Consider the successive terms, α'_j, in (13.24). Since V_{LP} always restricts supports to $[-\rho,\rho]$ in s,

$$\operatorname{supp}(\alpha'_j)\subseteq\{t\ge-\rho\}\cap\{s\ge-\rho\}\cap\{t-s\ge-2\rho\}\cap\{t-s\le2j\rho\}. \tag{13.36}$$

To get the expansion (13.20) we need to use (13.24) and then project each term with χ_ρ – after restricting to $s=\rho$, $\omega=-\theta$ (and shifting in t) to get the scattering kernel. Thus if

$$\kappa_j(s,\omega,\theta)=\alpha'_j(s-\rho,\rho,\theta,\omega) \tag{13.37}$$

then

$$\beta_j(V)=\chi_\rho[\kappa_j(s,-\theta,\theta)]. \tag{13.38}$$

Since, as a function of $t-s,s,\omega$ and θ α'_j is independent of s in $s>-\rho$ it follows from (13.35) that

$$\kappa_j\in H^{-\frac{n+1}{2}+\min(j,n+1)}([-2\rho,T)\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})\text{ for any T.} \tag{13.39}$$

Restricting to $\omega=-\theta$, a submanifold of codimension $n-1$ shows that

$$\kappa_j(s,-\theta,\theta)\in H^1([-2\rho,T)\times\mathbb{S}^{n-1})\text{ if }j\ge n+1. \tag{13.40}$$

Moreover, to get (13.40) we only use the regularity property (13.33) for the first $n+1$ factors in (13.31). Thus we conclude that the map

$$\dot H^{\frac{n+1}{2}}(B(\rho))\longrightarrow\sum_{j\ge n+1}\kappa_j(s,-\theta,\theta)\in H^1([-2\rho,T)\times\mathbb{S}^{n-1})\text{ is entire} \tag{13.41}$$

for each ρ. This is a good deal weaker than we need to prove Theorem 13.19. Obviously we need to examine the first $n+1$ terms in the taylor series of β at $V=0$ to show that this polynomial in V is defined and in any case we have to show that the whole map β takes values in H^2 rather than H^1. Nevertheless we shall use (13.41) because it allows us to prove that β is entire, with values in the good space (more or less because of Petit's theorem).

PROOF. Proof of Theorem 13.19 Starting at the beginning, consider

$$\kappa_1(s,-\theta,\theta)=\alpha_1'(s-\rho,\rho,-\theta,\theta). \tag{13.42}$$

This already has support in $[-2\rho,2\rho]$. We wish to show that this, the linear, term is as claimed in (13.21). We proceed to compute κ_1 explicitly. It is convenient to take the Fourier transform in s :

$$\widehat{k_1}(\lambda,\omega,\theta)=\int_{-\infty}^{\infty}e^{-i\lambda t}\kappa_1(t,\omega,\theta)dt=\widehat{\alpha_1}(\lambda,\rho,\omega,\theta)e^{i\lambda\rho}. \tag{13.43}$$

From the definition of α_1, this gives

$$\begin{aligned}\widehat{\kappa_1}(\lambda,\omega,\theta)&=e^{i\lambda\rho}\int_{-\infty}^{\infty}\int e^{-i\lambda(\rho-s')}\left[V_{\mathrm{LP}}e^{-i\lambda s}\delta_\theta(\omega)\right]ds'.\\ &=c_n^2\int e^{i\lambda s}D_s^{\frac{n-1}{2}}\int_{x\cdot\omega=s}V(x)\lambda^{\frac{n-3}{2}}e^{-i\lambda x\cdot\theta}dxds\end{aligned} \tag{13.44}$$

Integrating by parts we get

$$\widehat{\kappa_1}(\lambda,\omega,\theta)=c_n^2\lambda^{n-2}\int e^{i\lambda x\cdot(\omega-\theta)}V(x)dx. \tag{13.45}$$

Setting $\omega=-\theta$ we find

$$\widehat{\kappa_1}(\lambda,-\theta,\theta)=c_n^2\lambda^{n-2}\hat{V}(2\lambda\theta). \tag{13.46}$$

Thus $\widehat{\beta_1(V)}$ is the (n-dimensional) Fourier transform of $2^{-n}V(\frac{x}{2})=\tilde{V}$. Hence,

$$\beta_1=c_nD_s^{\frac{n-3}{2}}R_n\tilde{V} \tag{13.47}$$

shows that β_1 maps into $\dot{H}^2([-2\rho,2\rho]\times\mathbb{S}^{n-1})$. It is obviously an isomorphism onto $D_s^{\frac{n-3}{2}}R_n\dot{H}^{\frac{n+1}{2}}(B(2\rho))$ (which is closed) as claimed. □

We proceed to find a formula generalizing (13.45) to the higher derivatives at zero. From (13.36) we see that, for s bounded above, the support of each α_j' is compact in t. After taking the Fourier transform in t, the iterative definition (13.24) becomes:-

$$\widehat{\alpha_j'}(\lambda,s,\omega,\theta)=(D_s+\lambda)^{-1}R_n[V\cdot Q_\lambda]^{j-1}VR^tD_s^{(n-3)/2}e^{-is\lambda}\delta_\theta(\omega), \tag{13.48}$$

where

$$Q_\lambda=R_n^tD_s^{-1}(D_s+\lambda)^{-1}R_n. \tag{13.49}$$

Here D_s^{-1}, and $(D_s+\lambda)^{-1}$ mean integration from $s=-\infty$, i.e. the inverse preserving vanishing to the left.

LEMMA 13.46. *Acting from* $\mathcal{C}_c^\infty(\mathbb{R}^n)$ *to* $\mathcal{C}^\infty(\mathbb{R}^n)$, $Q_\lambda=(\Delta-\lambda^2)^{-1}$ *is the analytic extension of the 'free resolvent' defined as a bounded operator on* L^2 *for* $\Im\lambda<0$.

PROOF. This formula can be deduced from the modified Radon transform of Lax and Phillips. We know that this intertwines the wave group $U(t)$ with the translation group, so conjugates the infinitesimal generator of one to that of the other

$$c_n(D_s^{\frac{n-1}{2}}R, D_s^{\frac{n+1}{2}}R)\begin{pmatrix}0 & -1\\ \Delta & 0\end{pmatrix} = D_s(D_s^{\frac{n-1}{2}}R, D_s^{\frac{n+1}{2}}R). \tag{13.50}$$

From this we conclude that

$$c_n^2 R^t D_s^{\frac{n-3}{2}}(D_s+\lambda)^{-1}D_s^{\frac{n-1}{2}} = (\Delta-\lambda^2)^{-1}. \tag{13.51}$$

This proves the lemma. □

Inserting the integral expression for $(D_s+\lambda)^{-1}$ into (13.48) gives

$$\begin{gathered}\widehat{\alpha_j'}(\lambda, s, \omega, \theta) = \\ c_n^2 \int_{-\infty}^{s} e^{-i\lambda(s-s')}D_{s'}^{\frac{n-1}{2}} \int_{x\cdot\omega=s'} V\cdot Q_\lambda \cdot V \cdots \\ Q_\lambda \cdot [V(\bullet)(-\lambda)^{\frac{n-3}{2}}e^{-i\lambda\bullet\cdot\theta}]dH_x ds'.\end{gathered} \tag{13.52}$$

From (13.37), by setting $s=\rho$ and integrating by parts we get

$$\widehat{\kappa_j}(\lambda,\omega,\theta) = c_n^2(-1)^{\frac{n-3}{2}}\lambda^{n-2}\int_{\mathbb{R}^n} e^{i\lambda\omega\cdot x}V(x)[Q_\lambda\cdots Q_\lambda\cdot V(\bullet)e^{-i\lambda\theta\cdot\bullet}](x)dx. \tag{13.53}$$

Restricting to backscattering, $\omega=-\theta$, this gives $\widehat{\kappa_j}$ in a from similar to (13.46). Since κ_j has support in $[-2\rho, 2j\rho]$ its regularity can be deduced from its Fourier-Laplace transform with $\Im\lambda=-1$. Thus we need to examine the growth in λ of

$$\begin{gathered}\widehat{\kappa_j}(\lambda, -\theta, \theta) = \\ c_n^2\lambda^{n-2}\int_{\mathbb{R}^{jn}} e^{-i\lambda\theta\cdot(x^{(1)}+x^{(j)})}V(x^{(1)})Q_\lambda(x^{(1)}-x^{(2)})V(x^{(2)})\dots \\ \dots Q_\lambda(x^{(j-1)}-x^{(j)})V(x^{(j)})(x)dx^{(1)}\dots dx^{(j)}\end{gathered} \tag{13.54}$$

where there are $j-1$ factors of the free resolvent, Q_λ, and j factors of V. As a convolution operator Q_λ has kernel

$$Q_\lambda(y) = (2\pi)^{-n}\int e^{iy\cdot\eta}(|\eta|^2-\lambda^2)^{-1}d\eta. \tag{13.55}$$

Inserting this into (13.54) gives

$$\begin{gathered}\widehat{\kappa_j}(\lambda,-\theta,\theta) = \\ c_n^2\int e^{-i\xi\cdot(x^{(1)}+x^{(j)})}V(x^{(1)})V(x^{(2)})\dots V(x^{(j)})\prod_{\ell=1}^{j-1}(|\eta^{(\ell)}|^2-\lambda^2)^{-1} \\ \times\exp[i(x^{(1)}-x^{(2)})\cdot\eta^{(1)}+\cdots+i(x^{(j-1)}-x^{(j)})\cdot\eta^{(j-1)}] \\ dx^{(1)}\dots dx^{(j-1)}d\eta^{(1)}\dots d\eta^{(j-1)}\end{gathered} \tag{13.56}$$

where $\xi=\lambda\theta$.

Carrying out the x-integrals in (13.56) gives

(13.57)
$$\widehat{\kappa_j}(\lambda,-\theta,\theta)$$
$$= c_n^2\lambda^{n-2}\int \hat{V}(\xi+\eta^{(1)})\hat{V}(\eta^{(1)}-\eta^{(2)})\dots\hat{V}(\eta^{(j-2)}-\eta^{(j-1)})\hat{V}(\eta^{(j-1)}-\xi)$$
$$\prod_{\ell=1}^{j-1}(|\eta^{(\ell)}|^2-\lambda^2)^{-1}d\eta^{(1)}\dots d\eta^{(j-1)}.$$

Apart from the factors arising from the resolvent this is an iterated convolution. Since $\Im\lambda=-1$, the resolvent factors are non-singular. Using the obvious estimates

$$|(|\eta|^2-\lambda^2)^{-1}|\le c(1+|\eta|+|\lambda|)^{-1}. \tag{13.58}$$

and

$$(1+|\eta'|+|\lambda|)^{-1}(1+|\eta|+|\lambda|)^{-1}\le(1+|\eta-\eta'|)^{-1} \tag{13.59}$$

the right side of (13.57) can be estimated to give

$$|\widehat{\kappa_j}(\lambda,-\theta,\theta)|\le C^{j+1}|\lambda|^{n-2}\times$$
$$\int\hat{\Phi}(\xi+\eta^{(1)})\hat{\Phi}(\eta^{(1)}-\eta^{(2)}) \tag{13.60}$$
$$\dots\hat{\Phi}(\eta^{(j-2)}-\eta^{(j-1)})\hat{\Phi}(\eta^{(j-1)}+\xi)d\eta^{(1)}\dots d\eta^{(j-1)},$$

where

$$\hat{\Phi}(\eta)=|\hat{V}(\eta)|(1+|\eta|)^{-\frac{1}{2}}. \tag{13.61}$$

Thus

$$\|\Phi\|_{H^{(n+2)/2}}\le\|V\|_{H^{(n+1)/2}}. \tag{13.62}$$

First translating the variables of integration to $\eta^{(\ell)}+\xi$ we find that the right side of (13.60) is the Fourier transform of a product of functions, so using Lemma 13.44 repeatedly (and taking into account the factor of λ^{n-2})

$$\|\kappa_j(s,-\theta,\theta)\|_{H^{\frac{5}{2}}([-2\rho,2\rho]\times\mathbb{S}^{n-1}}\le C^{1+j}\|V\|_{H^{(n+1)/2}}. \tag{13.63}$$

This gives the desired continuity (13.22) and estimates (13.23) for $\epsilon=0$. Moreover the estimates (13.45) give (13.23) for $\epsilon=\frac{1}{2}$ and large (hence all) j. The estimates for all $\epsilon\in[0,\frac{1}{2}]$ then follow by interpolation between Sobolev spaces, i.e.

$$\|u\|_{\frac{5}{2}-\epsilon}\le C\|u\|_2^{2\epsilon}\|u\|_{\frac{5}{2}}^{1-2\epsilon}\quad\forall\ \epsilon\in[0,\frac{1}{2}]. \tag{13.64}$$

This completes the proof of Theorem 13.19.

Exercise 13.9. Can the estimates centred on (13.58) be improved to give the exponential type estimates (13.23) directly and with values in $H^{\frac{5}{2}}$?

Exercise 13.10. Show that if the original regularity $(n+1)/2$ for V is increased by p then the regularity of the derivatives β_j in (13.23) can also be increased by p.

PROPOSITION 13.36. *There is a closed subset of* $G(\rho) \subset \dot{H}^{\frac{n+1}{2}}(B(\rho))$ *which is of codimension at least two (i.e. locally orthogonal projection from* $G(\rho)$ *onto some subspace of codimension two is at most* p*-to-*1 *for some fixed* $p \in \mathbb{N}$*) such that for each* $V' \in [\dot{H}^{\frac{n+1}{2}}(B(\rho)) \setminus G(\rho)]$ *there exists* $\epsilon > 0$ *such that the map*

$$\beta_\rho : \left\{V \in \dot{H}^{\frac{n+1}{2}}(B(\rho)); \|V - V'\| < \epsilon\right\} \longrightarrow \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}) \tag{13.65}$$

is an isomorphism onto its image.

PROOF. The set $G(\rho)$ consists of those $V \in \dot{H}^{\frac{n+1}{2}}(B(\rho))$ such that the derivative of β_ρ with respect to V is not an isomorphism. Certainly (13.65) holds for points in the complement of $G(\rho)$ by the implicit function theorem. Thus we need to show that $G(\rho)$ so defined has codimension at least 2, since the density of the complement certainly follows form this. The derivative of β_ρ with respect to V is a linear map

$$\beta_1 + \gamma(V) : \dot{H}^{\frac{n+1}{2}}(B(\rho)) \longrightarrow \dot{H}^2([-2\rho, 2\rho] \times \mathbb{S}^{n-1}) \tag{13.66}$$

where β_1 is an isomorphism and $\gamma(V)$ depends analytically on V and maps continuously into $\dot{H}^{\frac{5}{2}-\epsilon}([-2\rho, 2\rho] \times \mathbb{S}^{n-1})$. If we consider simply the complex multiples of V, i.e. just look at $\gamma(zV)$, we have analyticity in z. The invertibility of this operator reduces to a finite dimensional problem, just as in the discussion surrounding (10.22). Thus invertibility can only fail at isolated values of z. This proves the result.

COROLLARY 13.6. *For each* $\rho > 0$ *there is a dense subset of* $\mathcal{C}^\infty(B(\rho))$ *near each point of which the backscattering transform* (13.2) *is injective from* $\mathcal{C}^\infty(B(\rho))$.

□

CHAPTER 14

Trace formulæ

We shall derive a trace formula, (14.73), expressing the trace of the wave group in terms of the determinant of the scattering operator. This formula is used to investigate the asymptotic behaviour of the determinant. There is another, closely related formula, (14.75), linking the poles of the scattering matrix with the regularized trace of the wave group. The two trace formula are used together in Chapter 15 to show that a non-vanishing potential must always have an infinite set of scattering poles.

To start we shall briefly consider operators of trace class. A neat and elegant treatment can be found in [**3**], Chapter 19. For simplicity we only discuss 'concrete' cases where the underlying space is

$$M = \mathbb{R}^n \times \mathbb{S}^p \times I^q,\ I \subset \mathbb{R} \text{ an interval,} \tag{14.1}$$

with any of the dimensions n, p and q permitted to be zero. The operators are then represented by their Schwartz kernels, as distributions on the product $M \times M$.

Trace class operators can be decomposed in terms of Hilbert-Schmidt operators, so we consider these first. The space of Hilbert-Schmidt operators is just

$$\mathrm{HS}(M) \equiv L^2(M \times M) \tag{14.2}$$

where the measure is the Lebesgue measure on $M \times M$. From the Cauchy-Schwarz inequality such an operator is bounded on $L^2(M)$. Thus

$$\begin{aligned} \|Bu\|^2 &= \int\limits_M |\int\limits_M B(x,x')u(x')dx'|^2 dx \\ &\le \int\limits_{M\times M} |B(x,x')|^2 dx dx' \int\limits_M |u(x')|^2 dx' \\ &\Longrightarrow \|B\|_{L^2(M)} \le \|B\|_{L^2(M\times M)}, \end{aligned} \tag{14.3}$$

bounds the operator norm in terms of the norm on $\mathrm{HS}(M)$. In fact it follows from this estimate that each Hilbert-Schmidt operator is compact on $L^2(M)$ since the kernel can be approximated in $\mathrm{HS}(M)$ by finite rank kernels, namely the expansion in any product orthonormal basis $\psi_j \otimes \phi_k$ for ψ_j and ϕ_k orthonormal bases of $L^2(M)$. We also note that $\mathrm{HS}(M)$ is a right ideal in the ring of all bounded operators on $L^2(M)$. Indeed, by Fubini's theorem $L^2(M \times M)$ is isomorphic to the space $L^2(M; L^2(M))$ of functions on M which are square-integrable with values in the Hilbert space $L^2(M)$. Since the kernel of $A \cdot B$, where B is a bounded operator on $L^2(M)$ and $A \in \mathrm{HS}(M)$, is just $B^t A(x, \bullet)$ it is also square-integrable with values in $L^2(M)$ so $A \cdot B \in \mathrm{HS}(M)$. Clearly $\mathrm{HS}(M)$ is invariant under the passage to transpose (or adjoint), so in fact is a two-sided ideal.

Since the condition (14.2) on the kernel of a Hilbert-Schmidt operator is so explicit it is easy to find examples. For instance

$$\mathcal{S}(\mathbb{R}^{2n}) \subset \mathrm{HS}(\mathbb{R}^n). \tag{14.4}$$

A useful example with finite regularity and growth is the operator

$$(1+|x|^2)^{s/2}(1+|D|^2)^{s/2} \in \mathrm{HS}(\mathbb{R}^n) \text{ iff } s < -\frac{n}{2}. \tag{14.5}$$

We shall say an operator is trace class if it is a finite sum of composites of pairs of Hilbert-Schmidt operators

$$T = \sum_{i=1}^{N} A_i \cdot B_i,\ A_i, B_i \in \mathrm{HS}(M) \Longleftrightarrow T \in \mathrm{TC}(M). \tag{14.6}$$

The space of trace class operators is clearly a two-sided ideal in the bounded operators. Suppose that ϕ_k and ψ_j are any orthonormal sequences in $L^2(M)$ then the series $\sum\limits_j \langle \psi_j, T\phi_j\rangle$ converges absolutely. This follows from (14.6):

$$\begin{aligned} \sum_j |\langle \psi_j, T\phi_j\rangle| &\le \sum_i \sum_j |\langle A_i^* \psi_j, B_i \phi_j\rangle| \\ \frac{1}{2} &\le \sum_i \sum_j [\|A_i^*\psi_j\|^2 + \|B_i\phi_j\|^2]. \end{aligned} \tag{14.7}$$

Then we can use the obvious estimate that for $A \in \mathrm{HS}(M)$ and any orthonormal sequence ϕ_j

$$\sum_j \|A\phi_j\|^2 \le \int\limits_{M\times M} |A(x,x')|^2 dx dx'. \tag{14.8}$$

EXERCISE 14.11. Check that the condition

$$\sum_i \|A\phi_i\|^2 < \infty \tag{14.9}$$

on a bounded operator, A, on $L^2(M)$ for any (one) orthonormal basis ϕ_i is equivalent to $A \in \mathrm{HS}(M)$.

The norm of an element $T \in \mathrm{TC}(M)$ is defined to be

$$\|T\|_{\mathrm{TC}} = \sup \sum_j |\langle \psi_j, T\phi_j\rangle| \tag{14.10}$$

where the supremum is over orthonormal bases.

EXERCISE 14.12. Show that with this norm $\mathrm{TC}(M)$ is a Banach space in which the finite rank operators form a dense subspace.

The convergence of the sum in (14.7) implies in particular that the trace of the element can be defined by

$$\mathrm{tr}(T) = \sum_j \langle T\phi_j, \phi_j\rangle,\ T \in \mathrm{TC}(M) \tag{14.11}$$

if ϕ_j is a complete orthonormal system in $L^2(M)$. Of course it is of the greatest importance that this limit is independent of the choice of orthonormal basis used to define it.

PROPOSITION 14.37. *For any $T \in \mathrm{TC}(M)$ the trace of T, defined by (14.11), is independent of the orthonormal basis used to define it.*

PROOF. We start by fixing the orthonormal basis ϕ_j and using (14.11) as the definition of $\mathrm{tr}(T)$. This is clearly a continuous linear functional on $\mathrm{TC}(M)$. Thus it suffices to prove the invariance of the choice of basis for finite rank operators. In fact, since it is linear in T, it is enough to consider operators of rank one, i.e. of the form

$$T = a \otimes b^*,\ a, b \in L^2(M) \text{ so } Tf = \langle f, b\rangle a. \tag{14.12}$$

In that case, using a familiar Hilbert space formula

$$\mathrm{tr}(T) = \sum_i \langle T\phi_i, \phi_i\rangle = \sum_i \langle \phi_i, b\rangle\langle a, \phi_i\rangle = \langle a, b\rangle \tag{14.13}$$

is independent of the choice of ϕ_i. This proves the independence in general. □

Note that if U is a unitary operator on $L^2(M)$ then

$$\mathrm{tr}(U^* \cdot T \cdot U) = \mathrm{tr}(T). \tag{14.14}$$

this just follows from the fact that $U\phi_j$ is another orthonormal basis. This identity can be rewritten $\mathrm{tr}(U \cdot T) = \mathrm{tr}(T \cdot U)$. More generally suppose that B is any bounded operator on $L^2(M)$ then

$$\mathrm{tr}(T \cdot B) = \mathrm{tr}(B \cdot T) \quad \forall\ T \in \mathrm{TC}(M). \tag{14.15}$$

Again this is true when T has rank one, since when T is of the form (14.12) then $B \cdot T = Ba \otimes b^*$ and $T \cdot B = a \otimes B^*b$. Using (14.13)

$$\mathrm{tr}(B \cdot T) = \langle Ba, b\rangle = \langle a, B^*b\rangle = \mathrm{tr}(T \cdot B). \tag{14.16}$$

By linearity and continuity (14.15) therefore holds in general. Another consequence of such arguments is an integral formula for the trace.

LEMMA 14.47. *If $T \in \mathrm{TC}(M)$ then the Schwartz kernel is, near the diagonal of $M \times M$, a continuous function of the difference $x - x'$ with values in $L^1(M)$ such that*

$$\mathrm{tr}(T) = \int_M T(x, x)dx. \tag{14.17}$$

PROOF. Assume for the moment that $M = \mathbb{R}^n$. If $A \in L^2(M \times M)$ then the continuity-in-the-mean of L^2 functions shows that

$$M \ni z \longmapsto A(x + z, y) \in L^2(M \times M) \tag{14.18}$$

is continuous. If $B \in L^2(M \times M)$ then

$$M \ni z \longmapsto A(x + z, y)B(y, x) \in L^1(M \times M) \tag{14.19}$$

is also continuous. Thus the kernel of $T = A \cdot B$ is such that

$$M \ni z \longmapsto T(x + z, x) = \int_M A(x + z, y)B(y, x)dy \in L^1(M) \tag{14.20}$$

is continuous. From (14.7) this proves the statement about the kernel for general trace class operators on $\mathbb{R}^n$. The other cases of M follow similarly.

The formula (14.17) now makes sense and its validity in general follows from the finite rank case by continuity. □

Next we note two simple conditions on the kernel of an operator sufficient to guarantee that it is of trace class.

LEMMA 14.48. *If* $T : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^{-\infty}(M)$ *is an operator with a kernel* $T \in H^s_c(\overset{\circ}{\rightarrow} M \times \overset{\circ}{\rightarrow} M)$ *for some* $s > \frac{1}{2}\dim(M)$ *then* $T \in \mathrm{TC}(M)$; *also* $\mathcal{S}(M \times M) \subset \mathrm{TC}(M)$.

PROOF. First just consider the case $M = \mathbb{R}^n$. Then the action of T can be written in terms of the Fourier transform:

$$\widehat{Tf}(\xi) = \int_{\mathbb{R}^n} \widehat{T}(\xi, \eta)\hat{f}(\eta)d\eta \tag{14.21}$$

with

$$\widehat{T}(\xi, \eta) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-ix\cdot\xi + iy\cdot\eta} T(x, y)dxdy. \tag{14.22}$$

The regularity assumption on the kernel of T means that

$$\widehat{T}(\xi, \eta)(1 + |\eta|^2)^s = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-ix\cdot\xi + iy\cdot\eta} A(x, y)dxdy,\ A \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{14.23}$$

Since $s > \frac{1}{2}n$ the function

$$G(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1 + |\xi|^2)^s d\xi \in L^2(\mathbb{R}^n). \tag{14.24}$$

Of course this does not mean that the convolution operator with kernel $G(x - y)$ is Hilbert-Schmidt. However, since the kernel of T has compact support we can choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $T(x, y)\phi(y) = T(x, y)$. Then $B(x, y) = G(x-y)\phi(y) \in \mathrm{HS}(\mathbb{R}^n)$ and $T = A \cdot B$ shows that T is trace class.

The same argument applies to the case that $M = \mathbb{R}^n \times I^q$ since the kernel is assumed to have compact support in the interior of the product. Moreover by localizing supports on the sphere the argument applies in case there are spherical factors too.

The second statement in the lemma follows by similar arguments, taking instead $\phi = (1 + |x|^2)^{-n}$. □

This discussion further extends to the case of matrices of operators on M; we shall not even change the notation. A matrix of operators is Hilbert-Schmidt (respectively trace class) if all its entries are Hilbert-Schmidt (resp. trace class). In particular we wish to consider the trace of the operator $U(t)$, which is a 2×2 matrix of operators on $M = \mathbb{R}^n$. This is certainly not trace class, since it is invertible and hence not even compact. The time averaged operator

$$U(\psi) = \int_{\mathbb{R}} \psi(t)U(t)dt,\ \psi \in \mathcal{C}^\infty_c(\mathbb{R}) \tag{14.25}$$

has a smooth kernel, but is still not trace class because the kernel is translation invariant near infinity (so has no decay properties, hence is not compact.) On the other hand the normalized operator

(14.26)
$$T(\psi) = U(\psi) - U_0(\psi) = \int_{\mathbb{R}} \psi(t)[U(t) - U_0(t)]dt \in \mathrm{TC}(\mathbb{R}^n) \quad \forall\ \psi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Indeed, the kernel is smooth and has compact support, so Lemma 14.48 applies. The first trace formula we shall prove expresses the trace of this operator in terms of the reduced scattering operator.

As usual we shall transfer the computation from physical space to the Radon transform side, where the free group $U_0(t)$ becomes particularly simple. To do so we shall use a slight extension of the invariance of the trace under conjugation. From (14.15) it follows that if B is an invertible linear map on $L^2(M)$ then

$$\mathrm{tr}(B^{-1} \cdot T \cdot B) = \mathrm{tr}(T) \quad \forall\ T \in \mathrm{TC}(M). \tag{14.27}$$

More generally if $B : L^2(M) \longrightarrow L^2(M')$ is a linear isomorphism then again it follows that $T' = B^{-1} \cdot T \cdot B \in \mathrm{TC}(M')$ for any $T \in \mathrm{TC}(M)$ and that $\mathrm{tr}(T') = \mathrm{tr}(T)$.

We wish to apply this identity to the modified Radon transform LP . The small difficulty is that LP is not an isomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$, but rather (8.75) holds, where $\mathcal{H}(\mathbb{R}^n)$ is the finite energy space. This space is given by (8.74) and can be written

$$\mathcal{H}(\mathbb{R}^n) = [|D|^{-1} L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n). \tag{14.28}$$

Here

$$\begin{gathered} f \in |D|^{-1} L^2(\mathbb{R}^n) \Longleftrightarrow \\ f \in \mathcal{S}'(\mathbb{R}^n),\ \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |\xi|\hat{f}(\xi) \in L^2(\mathbb{R}^n) \quad (n \geq 3). \end{gathered} \tag{14.29}$$

Now suppose $T \in \mathcal{S}(\mathbb{R}^{2n}) \subset \mathrm{TC}(\mathbb{R}^n)$. Consider the operators

$$\widetilde{T} = |D| \cdot T \cdot |D|^{-1},\ T_1 = |D| \cdot T \text{ and } T_2 = T \cdot |D|^{-1}. \tag{14.30}$$

Lemma 14.49. *If $T \in \mathcal{S}(\mathbb{R}^{2n})$ then T_1, T_2 and $\widetilde{T} \in \mathrm{TC}(\mathbb{R}^n)$ and*

$$\mathrm{tr}(T) = \mathrm{tr}(\widetilde{T}). \tag{14.31}$$

Proof. The operators $A = (1+|D|^2)^{-1}|D|$ and $B = (1+|x|^2)^{-2}(1+|D|^2)^{-n}|D|^{-1}$ are bounded on $L^2(\mathbb{R}^n)$, indeed B is Hilbert-Schmidt. Since $T = T'(1+|x|^2)^{-n}(1+|D|^2)^{-n}$ and $T = (1+|D|^2)^{-n}T''$ with $T', T'' \in \mathcal{S}(\mathbb{R}^n)$ all three operators defined from T are clearly trace class. The identity (14.31) follows from the fact that if $\widehat{T}$ is given by (14.22) then

$$\mathrm{tr}(T) = \mathrm{tr}(\mathcal{F} \cdot T \cdot \mathcal{F}^*) = \int_{\mathbb{R}^n} \widehat{T}(\xi, \xi) d\xi. \tag{14.32}$$

Similarly, since $\mathcal{F} \cdot \widetilde{T} \cdot \mathcal{F}^*$ has kernel $|\xi|\widehat{T}(\xi, \eta)|\eta|^{-1}$

$$\mathrm{tr}(\widetilde{T}) = \mathrm{tr}(\mathcal{F} \cdot \widetilde{T} \cdot \mathcal{F}^*) = \int_{\mathbb{R}^n} \widehat{T}(\xi, \xi) d\xi. \tag{14.33}$$

This completes the proof of the lemma. □

PROPOSITION 14.38. *The operator*

$$\widetilde{W}(\phi) = \int\limits_{\mathbb{R}} \phi(t)[W(t) - W_0(t)]dt \in \mathrm{TC}(\mathbb{R}\times\mathbb{R}^n) \quad \forall\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}) \tag{14.34}$$

and

$$\mathrm{tr}(\widetilde{W}(\phi)) = \mathrm{tr}(T(\phi)) \quad \forall\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}),\ V \in \mathcal{C}^\infty_c(\mathbb{R}^n) \tag{14.35}$$

where $T(\phi)$ *is given by* (14.26).

PROOF. We already know that $\widetilde{W}(\phi) \in \mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{R}\times\mathbb{S}^{n-1})$ so it is certainly trace class. Moreover since $W_V(t) = \mathrm{LP}\cdot U_V(t)\cdot\mathrm{LP}^*$ for all $V \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ it follows that

$$\widetilde{W}(\phi) = \mathrm{LP}\cdot U(\phi)\cdot \mathrm{LP}^*\,. \tag{14.36}$$

Now choose as an orthonormal basis of $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ $\{LP\Phi_{i,\ell}\}$ where $\ell = 0, 1$, and $i = 1, \ldots, \infty$ and $\Phi_{i,0} = |D|^{-1}\Phi_{i,1}$ for some orthonormal basis $\Phi_{i,0}$ of $L^2(\mathbb{R}^n)$. The orthonormality of $\mathrm{LP}\,\Phi_{i,\ell}$ follows from Proposition 8.21. Moreover

$$\begin{aligned}\mathrm{tr}(\widetilde{W}(\phi)) &= \sum_{\ell=0,1}\sum_{i=1}^{\infty}\langle \widetilde{W}(\phi)\,\mathrm{LP}\,\Phi_{i,\ell}, \mathrm{LP}\,\Phi_{i,\ell}\rangle_{L^2(\mathbb{R}\times\mathbb{S}^{n-1})}\\ &= \sum_{\ell=0,1}\sum_{i=1}^{\infty}\langle \mathrm{LP}^*\cdot T(\phi)\cdot\mathrm{LP}\,\Phi_{i,\ell}, \Phi_{i,\ell}\rangle_{\mathcal{H}(\mathbb{R}^n)}.\end{aligned} \tag{14.37}$$

Using (8.74) this becomes

$$\begin{aligned}\mathrm{tr}(\widetilde{W}(\phi)) &= \sum_{i=1}^{\infty}\langle \Delta[T(\phi)]_{0,0}\Phi_{i,0}, \Phi_{i,0}\rangle_{L^2(\mathbb{R}^n)} + \sum_{i=1}^{\infty}\langle [T(\phi)]_{1,1}\Phi_{i,1}, \Phi_{i,1}\rangle_{L^2(\mathbb{R}^n)}\\ &= \mathrm{tr}([T(\phi)]_{0,0}) + \mathrm{tr}([T(\phi)]_{1,1})\\ &= \mathrm{tr}(T(\phi)).\end{aligned} \tag{14.38}$$

Here we have used Lemma 14.49. This proves (14.35). □

Although the normalizing term, $W_0(t)$, in (14.34) is necessary to make $\widetilde{W}(\phi)$ trace class its presence makes the trace difficult to compute. Since this term is independent of V we can remove it by differentiation. Thus consider the variation, with respect to the potential, of the wave group. It has already been shown that $W_V(t)$ depends analytically on $V \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. The Schwartz kernel of $W_V(t)$ satisfies

$$\begin{aligned}(D_t + D_s + V_{\mathrm{LP}})W_V(t, s, \omega, s', \theta) &= 0\\ W_V(0, s, \omega, s', \theta) &= \delta(s - s')\delta_\theta(\omega).\end{aligned} \tag{14.39}$$

Differentiating with respect to V and applying the derivative to $\widetilde{V} \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ gives

$$\frac{d}{dz}W_{V+z\widetilde{V}}|_{z=0} = \dot{W}(\widetilde{V}) \tag{14.40}$$

satisfying

$$\begin{aligned}(D_t + D_s + V_{\mathrm{LP}})\dot{W}(t, s, \omega, s', \theta) &= -\widetilde{V}_{\mathrm{LP}}W(t, s, \omega, s', \theta)\\ \dot{W}(0, s, \omega, s', \theta) &= 0.\end{aligned} \tag{14.41}$$

This can easily be solved by duHamel's principle, so

$$\dot{W}(t) = -\int_0^t W_V(t-s)\cdot \widetilde{V}_{\mathrm{LP}}\cdot W_V(s)ds. \tag{14.42}$$

From this we can easily evaluate the derivative of $\operatorname{tr}\widetilde{W}(\phi)$ with respect to V at $\widetilde{V}$ as

$$\begin{aligned}\operatorname{tr}\dot{W}(\phi,\widetilde{V}) =& \frac{d}{dz}\operatorname{tr} W_{V+z\widetilde{V}}(\phi)|_{z=0}\\ =& -\operatorname{tr}\int_{\mathbb{R}}\phi(t)\int_0^t W_V(t-s)\cdot\widetilde{V}_{\mathrm{LP}}\cdot W_V(s)dsdt\\ =& -\operatorname{tr}\int_{\mathbb{R}} t\phi(t)W_V(t)\widetilde{V}_{\mathrm{LP}}dt.\end{aligned} \tag{14.43}$$

We wish to show that this can be re-expressed in terms of the scattering matrix and its variation. To do so recall the basic properties of the wave operators and the scattering operator itself. These operators can all be written in terms of the kernel of $W(t)$.

$$W_\pm(s,\omega,s',\theta) = W(s\pm\rho,\pm\rho,\omega,s',\theta) \tag{14.44}$$

$$W^\#_\pm(s,\omega,s',\theta) = W(-s'\pm\rho,s,\omega,\pm\rho,\theta) \tag{14.45}$$

$$\begin{aligned}K(s,\omega,\theta) &= W(s+\rho,\rho,\omega,-\rho,\theta)\\ K^\#(s,\omega,\theta) &= W(s-\rho,-\rho,\omega,\rho,\theta)\end{aligned} \tag{14.46}$$

Proposition 14.39. *If $V\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ has no bound states (in particular if $V(x)\ge 0$) then all six kernels in* (14.44) - (14.46) *define operators (by convolution in s in the case of K and $K^\#$) acting on $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ and $\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})$, with $W^\#_\pm$ and $K^\#$ respectively the inverses of $W_\pm$ and K and such that*

$$K = W_+\cdot W^\#_-. \tag{14.47}$$

Proof. The assumption that V is non-negative means in particular that there are no bound states so, following ???, we know that $W(t,s,\omega,s',\theta)$ is rapidly decreasing as $|t|\to\infty$ if s and s' are bounded. It follows that

$$W_\pm(s,\omega,s',\theta) = \lim_{t\to\pm\infty} W_0(-t)W(t)(s,\omega,s',\theta) \tag{14.48}$$

as a limit of tempered distributions. More precisely if $f\in\mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{S}^{n-1})$ then

$$W_\pm f = \lim_{t\to\infty} W_0(-t)W(t)f \text{ in } \mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1}). \tag{14.49}$$

Since the norm (8.82) is equivalent to the norm on $\mathcal{H}(\mathbb{R}^n)$ (when $V\ge 0$) we know that the operators $W(t)$ are uniformly bounded on $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$. Thus

$$\|W_\pm f\|_{L^2} \le \sup_t \|W_0(-t)\|\|W(t)\|\|f\| \le C\|f\| \tag{14.50}$$

extend to bounded operators on $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$. It is similarly clear that $W_\pm$ act continuously on $\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})$.

In fact essentially the same arguments can be applied to $W_\pm^\#$ which are also strong limits of operators:

$$W_\pm^\# f = \lim_{t\to\pm\infty} W(-t)W_0(t). \tag{14.51}$$

These are obviously the inverses of $W_\pm$.

The fact that $W_\pm$ are invertible on $\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})$ can be translated to the statement that the boundary problems

$$\begin{aligned}(D_t + D_s + V_{\text{LP}})u =&0,\\ u_{|s=\pm\rho} =&f \in \mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1}).\\ u_{|t=0} \in&\mathcal{S}(\mathbb{R}\times\mathbb{S}^{n-1})\end{aligned} \tag{14.52}$$

each have a unique solution. The definition of the scattering operator then reduces to (14.47). □

Using the invariance of the trace under conjugation we can transform (14.43) to

$$\dot{W}(\phi,\widetilde{V}) = -\operatorname{tr}\int_{\mathbb{R}}[t\phi(t)]W_-\cdot W(t)\cdot \widetilde{V}_{\text{LP}}\cdot W_-^\# dt \quad (V\geq 0). \tag{14.53}$$

From (14.44) it follows that for any $t\in\mathbb{R}$

$$W_\pm\cdot W(t) = W_0(t)\cdot W_\pm. \tag{14.54}$$

Thus (14.53) becomes

$$\dot{W}(\phi,\widetilde{V}) = -\operatorname{tr}\int_{\mathbb{R}}[t\phi(t)]W_0(t)\cdot W_-\cdot \widetilde{V}_{\text{LP}}\cdot W_-^\# dt \quad (V\geq 0). \tag{14.55}$$

Using (14.47) this can be further rewritten as

$$\begin{aligned}\dot{W}(\phi,\widetilde{V}) &= -\operatorname{tr}\int_{\mathbb{R}}[t\phi(t)]K^\#\cdot G\cdot W_0(t)dt\\ G &= W_+\cdot\widetilde{V}_{\text{LP}}\cdot W_-^\#\end{aligned} \quad (V\geq 0). \tag{14.56}$$

The kernel of $G\cdot W_0(t)$ is just $G(s-t,\omega,s',\theta)$. In terms of the kernels (14.56) is therefore

$$\dot{W}(\phi,\widetilde{V}) = -\int[t\phi(t)]K^\#(s-s',\omega,\theta)G(s'-t,\theta,s,\theta)dtdsds'd\omega d\theta \quad (V\geq 0). \tag{14.57}$$

After integration in t the kernel is smooth and rapidly decreasing in s and s'. We can apply Plancherel's identity in the t variable and so get

$$\dot{W}(\phi,\widetilde{V}) = -\frac{1}{2\pi}\int_{\mathbb{R}}\frac{d\widehat{\phi}}{d\lambda}(\lambda)\dot{S}(\lambda)d\lambda \quad (V\geq 0) \tag{14.58}$$

where

$$\begin{aligned}&\dot{S}(\lambda) =\\ -i\int e^{i\lambda t}&\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} K^\#(s-s',\omega,\theta)G(s'-t,\theta,s,\omega)dtdsds'd\omega d\theta.\end{aligned} \tag{14.59}$$

Applying Plancherel's formula to (14.59) allows it to be written

$$\dot{S}(\lambda) = -i \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \widehat{K^{\#}}(\lambda,\omega,\theta)\widehat{H}(\lambda,\theta,\omega)d\lambda d\omega d\theta \tag{14.60}$$

with

$$H(s,\omega,\theta) = \int\limits_{\mathbb{R}} G(r,\omega,s-r,\theta)dr. \tag{14.61}$$

As we shall see below, provided $V \geq 0$, the operator $\widehat{H}(\lambda)$ has a smooth kernel on the sphere, for each $\lambda \in \mathbb{R}$. We have already shown $\widehat{K^{\#}}$ to be of the form $\operatorname{Id} + A_V(\lambda)$ with $A_V(\lambda)$ a smoothing operator on the sphere. Thus we can interpret (14.60) in terms of the trace of operators on $L^2(\mathbb{S}^{n-1})$:

$$\dot{S} = -i \operatorname{tr}[\widehat{K^{\#}}(\lambda)\cdot\widehat{H}(\lambda)] \quad (V \geq 0). \tag{14.62}$$

So we need to see how H is related to the scattering operator.

Lemma 14.50. *If $V \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$ and $V \geq 0$ then the variation of the scattering operator with respect to V is*

$$\frac{dK_{V+z\widetilde{V}}}{dz}|_{z=0} = H \tag{14.63}$$

given by (14.56) *and* (14.61).

Proof. We can deduce (14.63) from the variational formula (14.42) for the wave group and (14.47). It is slightly more convenient to combine (14.49), (14.51) and (14.47) to get

$$K = \lim_{t\to\infty} W_0(-t)W(2t)W_0(-t) \tag{14.64}$$

where again the limit is locally uniform. Differentiating (14.64) gives

$$\begin{aligned}
\dot{K} &= -\lim_{t\to\infty} W_0(-t)\int\limits_0^{2t} W(2t-r)\cdot\widetilde{V}_{\mathrm{LP}}\cdot W(r)drW_0(-t) \\
&= \lim_{t\to\infty}[W_0(-t)W(t)]\cdot\int\limits_{-t}^{t} W(-r)\cdot\widetilde{V}_{\mathrm{LP}}\cdot W(r)dr\cdot W(t)W_0(-t) \\
&= W_+\cdot\int\limits_{-\infty}^{\infty} W(r)\cdot\widetilde{V}_{\mathrm{LP}}\cdot W(r)dr\cdot W_-^{\#} \\
&= \int\limits_{-\infty}^{\infty} W_0(r)\cdot[W_+\cdot\widetilde{V}_{\mathrm{LP}}\cdot W_-^{\#}]W_0(r)dr.
\end{aligned} \tag{14.65}$$

The operator on the right here is precisely H. □

Now we can combine the formulæ (14.62) and (14.63) to conclude that (14.58) holds with

$$\dot{S}(\lambda) = -i\int\limits_0^1 \operatorname{tr}[\widehat{K^{\#}}_{rV}(\lambda)\cdot\frac{d}{dr}\widehat{K}_{rV}(\lambda)]dr \quad (V \geq 0). \tag{14.66}$$

To further simplify the form of this we consider the Fredholm determinant. Suppose that $T \in \mathrm{TC}(M)$ then we define the determinant of $\mathrm{Id} - zT$ for all $z \in \mathbb{C}$ by

$$\det(\mathrm{Id} - zT) = \exp\{\int_0^z \mathrm{tr}[(\mathrm{Id} - zT)^{-1} \cdot T] dz\}. \tag{14.67}$$

PROPOSITION 14.40. *The Fredholm determinant* $\det(\mathrm{Id} - zT)$, *defined by* (14.67) *for* $T \in \mathrm{TC}(M)$, *is an entire function of* $z \in \mathbb{C}$ *with zeros precisely at those points* z *such that* $1/z$ *is in the spectrum of* T, *the order of the zero is equal to the algebraic multiplicity of* $1/z$ *as an eigenvalue. If* $T, T' \in \mathrm{TC}(M)$ *then*

$$\det(\mathrm{Id} - T)\det(\mathrm{Id} - T') = \det(\mathrm{Id} - T''),\ T'' = T + T' + T \cdot T' \tag{14.68}$$

and if T *is quasi-nilpotent (has no non-zero eigenvalues) then* $\det(\mathrm{Id} - T) = 1$.

PROOF. Certainly (14.67) defines $\det(\mathrm{Id} - zT)$ as a holomorphic function near the origin. Indeed only the poles of the integral could prevent it being entire, since the trace is certainly meromorphic. First we wish to show that the determinant extends to be single-valued on the complement of the discrete set formed by the inverses of the eigenvalues of T. This follows from the fact that the residue of the exponent at each singular point is in $2\pi i\mathbb{Z}$. To see this, and indeed to check the analyticity of the determinant, observe that (14.67) is valid on a finite dimensional space – this is the standard formula for the derivative of a determinant. If λ is any non-zero eigenvalue of T then T can be decomposed into a sum of a finite rank operator, acting as T on the generalized eigenspace associated to λ, and a trace class operator with no eigenvalue at λ. Since the exponent in (14.67) splits as a sum under this decomposition, the determinant splits as a product and the holomorphy and vanishing statements are direct consequences of these same statements in the finite dimensional case.

Again from (14.67) and this discussion it follows that, locally uniformly in z, the determinant $\det(\mathrm{Id} - zT)$ depends continuously, even smoothly, on $T \in \mathrm{TC}(M)$. Since (14.68) holds for finite rank operators it therefore holds in general.

Finally then consider the triviality of the determinant for quasi-nilpotent operators. This is based on the fact that for any trace class operator and each $\epsilon > 0$ there exists C_ϵ such that

$$|\det(\mathrm{Id} - zT)| \le C_\epsilon \exp(\epsilon|z|)\ \forall\ z \in \mathbb{C}. \tag{14.69}$$

If T is quasi-nilpotent then it follows from the first part of the proposition that $\det(\mathrm{Id} - zT)$ has no zeros. Thus $f(z) = \log\det(\mathrm{Id} - zT)$ is entire and

$$|\Re f(z)| \le C_1 + \epsilon|z| \quad \forall\ z \in \mathbb{C}. \tag{14.70}$$

From this it follows that $f(z)$ is constant, hence $\det(\mathrm{Id} - zT)$ is equal to its value, 1, at 0. The estimate (14.69) is a form of Weyl's convexity estimates. □

Using the formula for the determinant we can rewrite (14.66) as

$$\mathrm{tr}\, T_V(\phi) = -\frac{1}{2\pi}\int \frac{d\widehat{\phi}}{d\lambda}(\lambda)S(\lambda)d\lambda,\ S(\lambda) = \frac{1}{i}\log\det[\widehat{K}_V(\lambda)] \quad (V \ge 0). \tag{14.71}$$

This is the trace formula we have been seeking, since it relates the normalized trace of the wave group to the determinant of the scattering matrix. The restriction to $V \ge 0$ (really V with no 'bound states') is a real one. The general formula is a little more complicated because $\det[\widehat{K}(\lambda)]$ can have singularities. Since $\widehat{K}(\lambda)$ can have poles on the real axis we shall define the scattering phase as a principal value:

$$S(\lambda) = \frac{1}{2i}\{\lim_{\Im\lambda\uparrow 0} + \lim_{\Im\lambda\downarrow 0}\} \log\det[\widehat{K}(\lambda)] \in \mathcal{S}'(\mathbb{R}). \tag{14.72}$$

The branch of the logarithm is fixed by requiring that it be the principal branch on the imaginary axis and holomorphic in a small split strip $0 < |\Im\lambda| < \epsilon$ for some $\epsilon > 0$. Of course the choice of the branch is not very significant since S is differentiated in the formula.

Theorem 14.20. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $n \geq 3$ odd, the normalized trace of the wave group can be written in terms of the scattering matrix as*

$$\begin{gathered} \operatorname{tr}\int_{\mathbb{R}} \phi(t)[U_V(t) - U_0(t)] = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{d\widehat{\phi}}{d\lambda}S(\lambda)d\lambda \\ +\frac{1}{2}\sum_{\Im\lambda_i=0}[\widehat{\phi}(\lambda_i)+\widehat{\phi}(-\lambda_i)]\operatorname{mult}(\lambda_i) + \sum_{\Im\lambda_i<0}[\widehat{\phi}(\lambda_i)+\widehat{\phi}(-\lambda_i)]\operatorname{mult}(\lambda_i) \\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \end{gathered} \tag{14.73}$$

where $\operatorname{mult}(\lambda)$ *is the algebraic multiplicity of λ as a pole of the scattering matrix.*

Proof. This is of course exactly (14.71) in case $V \geq 0$, or even if there are no poles in the (closed) lower half plane. To prove it for every V all we need to do is to show that the right side is an entire function of V. Once this is proved the validity of (14.71), which is the same formula for $V \geq 0$, implies that all derivatives, with respect to V at $V = 0$, of the two sides are equal when evaluated on non-negative potentials. Since the derivatives are polynomial in V, the fact that the non-negative potentials span $\mathcal{C}_c^\infty(\mathbb{R}^n)$ over $\mathbb{C}$ implies that (14.73) holds in general.

Thus we consider the holomorphy of the right side on (14.73) in V. This is a local condition. For any particular V there are no poles of $\widehat{K}_V(\lambda)$ in some split strip $0 < |\Im\lambda| < \epsilon$, with $\epsilon > 0$. Since $S(\lambda)$ is defined as a boundary value on the real axis from above and below the real integral in (14.73) can be shifted to a pair of contour integrals on $\Im\lambda = \pm\frac{1}{2}\epsilon$. Once so moved each is locally holomorphic in V. Similarly the two sums in (14.73), when rewritten as the sums, respectively, over the poles in $|\Im\lambda| < \frac{1}{2}\epsilon$ and $\Im\lambda < -\frac{1}{2}\epsilon$ are also locally analytic in V – since they can be written as a contour integrals involving the resolvent of $D_s + V_{\mathrm{LP}}$. It therefore only remains to note that this analytic extension of the expression on the right in (14.73) is equal to the right side near the fixed potential V. In fact this is just a computation of residues. Moving the two contour integrals back from $\Im\lambda = \pm\frac{1}{2}\epsilon$ to the real axis (or rather limits from above and below) gives the first term in (14.73) plus the residues, which occur at both poles and zeros of $\det(\widehat{K}(\lambda))$. Since the zeros are precisely the negatives of the poles, with multiplicities, these terms combine with the sum over the poles in $\lambda < -\frac{1}{2}\epsilon$ to give the correct sums as well. □

As noted earlier one way of viewing (14.73) is as a formula for the scattering phase, in terms of the more computable wave group.

Proposition 14.41. *For each $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the determinant* $\det(\widehat{K}(\lambda))$ *is a meromorphic function in $\mathbb{C}$ which, for each $c > 0$, has no poles in $|\Re(\lambda)| > C(c)$*

and is such that the logarithm has an asymptotic expansion of the form

$$\log\det(\widehat{K}(\lambda)) \sim \sum_{j=2}^{\infty} H_j(V)\lambda^{n-j} \text{ as } |\Re(\lambda)| \to \infty. \tag{14.74}$$

PROOF. □

The constants $H_j(V)$ are (non-linear) functionals of V known as the 'heat invariants.' They can be computed rather explicitly.

The second trace formula can be obtained from (14.73) by formally evaluating the right side by residues, shifting the contours to imaginary infinity and then ignoring them. We do not have anywhere near enough information to justify this directly, so we need a different approach, still we finally get:

THEOREM 14.21. *For any $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$*

$$\operatorname{tr}\int_0^\infty \phi(t)[U_V(t) - U_0(t)]dt = \sum \cos(t\lambda_j)\widehat{\phi}(\lambda_j)\operatorname{mult}(\lambda_j) \quad \forall\ \phi \in \mathcal{C}_c^\infty((0,\infty)). \tag{14.75}$$

The first component of the proof is Lidsky's theorem:

PROPOSITION 14.42. *If $T \in \mathrm{TC}(M)$ then the sequence of non-zero eigenvalues of T, repeated with multiplicity, is absolutely convergent and*

$$\operatorname{tr}(T) = \sum \operatorname{mult}(\lambda_j)\lambda_j. \tag{14.76}$$

PROOF. The non-zero eigenvalues have finite multiplicity so we can choose a basis of the generalized eigenspace with eigenvalue λ_j, $e_{1j}, \dots, e_{N_j j}$, N_j being the dimension of the eigenspaces, such that

$$[T - \lambda_j]e_{ij} \in \operatorname{sp}\{e_{\ell j}, \ell < j\}. \tag{14.77}$$

Applying the Gramm-Schmidt orthonormalization procedure we obtain ϕ_{ij} such that

$$\phi_{ij} = \alpha e_{ij} + \sum_{\ell < i}\beta_\ell e_{\ell j} + \sum_{k<j}\gamma_{\ell k}e_{\ell k}. \tag{14.78}$$

Clearly then

$$\langle T\phi_{ij}, \phi_{ij}\rangle = \lambda_j. \tag{14.79}$$

Thus the convergence of the sum $\sum |\lambda_j| \operatorname{mult}(\lambda_j)$ follows from (14.10).

Let Φ be the closure of the span of the eigenspaces associated to non-zero eigenvalues of T, i.e. the span of the ϕ_{ij}. Certainly $T\Phi \subset \Phi$. Similarly let Φ^* be the same space for T^* and let Φ^o be its orthocomplement. This splits the space

$$L^2(M) = \Phi + \Phi^o, \tag{14.80}$$

although the decomposition is not in general orthogonal. It follows that $T(\Phi^o) \subset \Phi^o$ and that

$$T'\phi = T\phi_2,\ \phi = \phi_1 + \phi_2,\ \phi_1 \in \Phi,\ \phi_2 \in \Phi^o \tag{14.81}$$

defines a quasi-nilpotent operator. Choosing an appropriate basis we see that

$$\operatorname{tr}(T) = \operatorname{tr}(T - T') + \operatorname{tr}(T'). \tag{14.82}$$

From (14.79) it follows that

$$\operatorname{tr}(T - T') = \sum \lambda_j \operatorname{mult}(\lambda_j). \tag{14.83}$$

Thus it remains only to show that $\operatorname{tr}(T') = 0$. This however follows from the last part of Proposition 14.40. □

PROOF. Proof of Theorem 14.21 To prove (14.75) we first write the trace in terms of the Lax-Phillips semigroup:

$$\operatorname{tr} \int_0^\infty \phi(t)[U_(t) - U_0(t)]dt = \operatorname{tr} \int_0^\infty [Z(t) - Z_0(t)]dt \quad \forall\ \phi \in \mathcal{C}_c^\infty((0,\infty)). \tag{14.84}$$

This is an immediate consequence of (14.35) and the fact that the restriction to the diagonal in $\mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-1}$ of the kernel of $\widetilde{W}(\phi)$ vanishes in the region $|s| \geq \rho$. The trace in (14.84) is over $M = [-\rho, \rho] \times \mathbb{S}^{n-1}$.

For each fixed $\phi \in \mathcal{C}_c^\infty((0,\infty))$ there exists $\epsilon = \epsilon_\phi > 0$ such that

$$\operatorname{supp}(\phi) \subset [\epsilon, \infty),\ 2\rho = q\epsilon,\ q \in \mathbb{N}. \tag{14.85}$$

We shall assume the stronger condition

$$\operatorname{supp}(\phi) \subset [\epsilon, 2\epsilon] \tag{14.86}$$

and later recover the general case by a finite decomposition. We then divide M into q pieces:

$$\begin{gathered} M = \bigcup_{j=1}^{q} M_j \text{ where} \\ M_j = [-\rho + (j-1)\epsilon, -\rho + j\epsilon] \times \mathbb{S}^{n-1} = M_j. \end{gathered} \tag{14.87}$$

Of course, each of these subsets is of the form (14.1). Let $\tau_i(s,\omega) = (s+\rho-(i-1)\epsilon, \omega)$ be the translation from M_i onto the fixed model $M_\epsilon = [0,\epsilon] \times \mathbb{S}^{n-1}$. Now we can regard an operator such as $\widetilde{W}(\phi)$ as a matrix of operators on $L^2(M_\epsilon)$:

$$\widetilde{W}(\phi)_{ij} = \pi_i \cdot \widetilde{W}(\phi) \cdot \iota_j,\ i,j = 1,\dots,q \tag{14.88}$$

where π_i is the orthogonal projection onto $L^2(M_i)$ from $L^2(M)$, followed by the pull back under τ_i, and ι_i is pull-back under the inverse of τ_i followed by the inclusion of $L^2(M_i)$ into $L^2(M)$ as a closed subspace by extending the elements as zero in $M \setminus M_i$. Since the kernel of each $\widetilde{W}(\phi)_{ij}$ is $\mathcal{C}^\infty$ these are all trace class operators and

$$\operatorname{tr}(\widetilde{W}(\phi)) = \sum_{i=1}^{q} \operatorname{tr}(\widetilde{W}(\phi)_{ii}). \tag{14.89}$$

Next we alter each of these 'components' of $\widetilde{W}(\phi)$ by conjugation with a power of an invertible elliptic operator. We set

$$\Gamma_\pm = (\varDelta_{\mathbb{S}^{n-1}} + 1)^{\pm n}. \tag{14.90}$$

Consider the matrix of operators

$$P_{ij} = \Gamma_-^i \cdot [\widetilde{W}(\phi)_{ij}] \cdot \Gamma_+^j. \tag{14.91}$$

These operators all have smooth kernels, so are trace class. Moreover,

$$\operatorname{tr}(P) = \operatorname{tr}(\widetilde{W}(\phi)). \tag{14.92}$$

Indeed the proof follows that of Lemma 14.49, using (14.89), the corresponding expansion for $\operatorname{tr}(P)$ and a basis of $L^2(M_i)$ consisting of eigenfunctions for $\Delta_{\mathbb{S}^{n-1}}$, i.e. spherical harmonics.

Now $\widetilde{W}(\phi)$ is the difference

$$\widetilde{W}(\phi) = Q_V - Q_0 \text{ where} \\ Q_V = \int_0^\infty \phi(t) W_V(t) dt. \tag{14.93}$$

Let

$$(Q_V)_{ij} = \pi_i \cdot Q_V \cdot \iota_j \text{ and } (Q'_V)_{ij} = \Gamma_-^i \cdot (Q_V)_{ij} \cdot \Gamma_+^j \tag{14.94}$$

be the corresponding decompositions analogous to (14.87) and (14.91). Clearly

$$P_{ij} = (Q_V)_{ij} - (Q_0)_{ij} \quad \forall\ i,j = 1,\dots,q. \tag{14.95}$$

The point of this construction is that now each $(Q_0)_{ij} \in \operatorname{TC}(M_\epsilon)$. To see this simply note that (14.86) means

$$(Q_0)_{ij} \pi_i \cdot \int_0^\infty \phi(t) W_0(t) dt \cdot \iota_j = \begin{cases} 0 & \text{if } i \neq j+1 \\ \Gamma_- \otimes \phi(\bullet - \epsilon)* & \text{if } i = j+1. \end{cases} \tag{14.96}$$

Thus in all cases, $(Q_0)_{ij}$ is trace class, so the matrix of operators is itself trace class.

Not only is Q_0 trace class but

$$\operatorname{tr}(Q_0) = \sum_{i=1}^q \operatorname{tr}((Q_0)_{ii}) = 0 \tag{14.97}$$

since the diagonal entries in the matrix are themselves zero. It follows that, for any V, $Q_V = P + Q_0$ is also trace class and

$$\operatorname{tr}(\widetilde{W}(\phi)) = \operatorname{tr}(P) = \operatorname{tr}(Q_V) = \sum \lambda'_j \operatorname{mult}_Q(\lambda'_j) \tag{14.98}$$

where $\operatorname{mult}_Q(\lambda)$ is the multiplicity of λ as an eigenvalue of Q_V.

The final step in the proof of (14.75), with ϕ restricted by (14.86), is to check that

$$\text{the non-zero eigenvalues of } Q_V \text{ are the non-zero values of} \\ \int_0^\infty \cos(t\lambda_j) \phi(t) dt \tag{14.99}$$

repeated, of course, with the multiplicity of λ_j as a pole of the scattering matrix and over those λ_j for which the integral takes on a fixed value.

From the discussion of the Lax-Phillips semigroup we do know that these are precisely the non-zero eigenvalues of

$$Z_V(\phi) = \int_0^\infty \phi(t) Z_V(t) dt. \tag{14.100}$$

Thus we only have to check that $Z_V(\phi)$ and Q_V have the same non-zero eigenvalues. This is the case because they are conjugate, although not by a bounded operator on $L^2(M)$. Consider some non-zero eigenvalue, μ, of Q_V, and the associated generalized eigenspace E. Thus $e \in E$ is a q-tuple of square-integrable functions, $e_i \in L^2(M_i)$. Now each component of P is a smoothing operator and each component of Q_0 is, by (14.96), smoothing of order n at least. Thus each component of $Q_V = P + Q_0$ must also be smoothing of order n. It follows that each component of any element of the eigenspace must be $\mathcal{C}^\infty$ (in all variables). Thus

$$f(s,\omega) = \sum_{i=1}^{q} [\iota_i \Gamma_+^i(e_i)](s,\omega) \in L^2([-\rho,\rho] \times \mathbb{S}^{n-1}). \tag{14.101}$$

From the definition of Q_V it is clear that as e ranges over E these functions span the corresponding eigenspace of $Z_V(\phi)$. Since this argument is easily reversed this proves (14.99).

Thus we have proved (14.75) when ϕ is subject to (14.86). Since the equation is linear and (14.86) can always be arranged by a finite decomposition this completes the proof of Theorem 14.21. □

CHAPTER 15

Scattering poles

We should include:

1) Various characterizations, symmetry 2) The logarithmic gap 3) Upper bound on the counting function 4) Lower bound following Sjöstrand and Zworski(?) 5) Some discussion of continuity(?) 6) Absence of poles implies $V \equiv 0$

CHAPTER 16

Hamilton-Jacobi theory

Next we turn to the scattering theory for a metric perturbation of the normal Laplacian on $\mathbb{R}^n$; again restricting ourselves to the case $n \geq 3$ odd. We shall consider Riemann metrics

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j \tag{16.1}$$

which differ from the flat metric only on a compact set,

$$g_{ij}(x) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{in } |x| > \rho. \tag{16.2}$$

Of course we require that g be positive definite:

$$\sum_{i,j=1}^{n} g_{ij}(x) \xi^i \xi^j \geq c|\xi|^2 \quad \forall\ \xi \in \mathbb{R}^n \text{ for some } c > 0. \tag{16.3}$$

Let $g^{ij}(x) = (g_{ij}(x))^{-1}$ be the inverse matrix and $g = \det g_{ij}$ the determinant. Then the Laplacian is

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \sqrt{g} g^{ij} \frac{\partial}{\partial x_j}. \tag{16.4}$$

Our immediate aim is to show the unique solvability of the Cauchy problem for the wave operator

$$P = D_t^2 - \Delta_g. \tag{16.5}$$

We shall do this much as in the case of a potential perturbation by constructing "plane wave" solutions.

Thus we look for a solution to

$$\begin{aligned} &Pu = (D_t^2 - \Delta_g)u \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n) \\ &u|_{t=0} = 0 \\ &D_t u|_{t=0} = \delta(s - x \cdot \omega). \end{aligned} \tag{16.6}$$

In the potential case we found the solution as a sum of conormal distributions associated to the two hypersurface $t = \pm(s - x \cdot \omega)$. For the metric problem the geometry itself has been perturbed so the first problem is to find the hypersurfaces with respect to which the solution should be conormal.

Suppose that we assume as an 'ansatz' that the solution to (16.6) is of the form

$$u = a_+ H(\phi_+(t, x, s, w)) + a_- H(\phi_-(t, x, s, w)) \tag{16.7}$$

where $a_\pm$ are $\mathcal{C}^\infty$ and $\phi_\pm$ are, at least for small $|t|$, $\mathcal{C}^\infty$ real functions with

$$\phi_\pm|_{t=0} = s - x\cdot\omega. \tag{16.8}$$

Thus the solution is supposed to be conormal with respect to $\{\phi_+ = 0\}\cup\{\phi_- = 0\}$.

Applying P directly to (16.7) we see that

$$\begin{aligned} Pu =& c_1^+\delta'(\phi_+) + c_1^-\delta'(\phi_-) + c_2^+\delta(\phi_+) \\ &+ c_2^-\delta(\phi_-) + c_3^+H(\phi_+) + c_3^-H(\phi_-), \end{aligned} \tag{16.9}$$

with all the coefficients $\mathcal{C}^\infty$. In fact we can easily compute the leading coefficients:

$$c_1^\pm = a_\pm\cdot\left[(D_t\phi_\pm)^2 - \sum_{i,j=1}^n g^{ij}(x)D_i\phi_\pm D_j\phi_\pm\right]. \tag{16.10}$$

In order for (16.6) to hold the leading terms in (16.9) must vanish – these being the most singular. Since $a_\pm$ should be nonzero this leads us to the *eikonal equation*

$$(D_t\phi)^2 - \sum_{i,j=1}^n g^{ij}(x)D_i\phi D_j\phi = 0 \text{ on } \phi = 0 \quad (\phi = \phi_\pm). \tag{16.11}$$

This may seem strange, that the search for a solution to a linear second order equation should lead to a non-linear first order equation, but the beauty of (16.11) is that it can be solved essentially geometrically by 'Hamilton-Jacobi theory.' We drop the restriction to $\{\phi = 0\}$ in (16.11) and try to solve the initial value problem:

$$\begin{aligned} (D_t\phi)^2 - \sum_{i,j=1}^n g^{ij}(x)D_i\phi D_j\phi = 0 \quad &\text{in } \Omega \\ \phi|_{t=0} = s - x\cdot\omega \qquad &\text{in } \Omega\cap\{t=0\} \end{aligned} \tag{16.12}$$

for Ω some neighbourhood of $0\in\mathbb{R}^N = \mathbb{R}^{n+1}$.

In the cotangent bundle $T^*\mathbb{R}^N = \mathbb{R}^N\times\mathbb{R}^N$ consider the graph of the differential of ϕ :

$$\Lambda_\phi = \left\{(z, d_z\phi)\in\mathbb{R}^N\times\mathbb{R}^N; z = (t,x), d_z\phi = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x_1}, \dots, \frac{\partial\phi}{\partial x_n}\right)\right\}. \tag{16.13}$$

Thus Λ_ϕ is an N-dimensional smooth submanifold of $T^*\mathbb{R}^N$. In terms of the 'canonically dual' coordinates (z,ζ) in $T^*\mathbb{R}^N$ (where an element of $T^*_x\mathbb{R}^N$ is written $\zeta\cdot dz$) consider the function

$$p(z,\zeta) = \zeta_0^2 - \sum_{i,j=1}^n g^{ij}(x)\zeta_i\zeta_j. \tag{16.14}$$

This is the principal symbol of the operator P. The differential equation (16.12) can then be written in geometric form as

$$\Lambda_\phi\subset\{p=0\} = \Sigma_p. \tag{16.15}$$

The surface Σ_p is called the *characteristic variety* of P.

Thus we have reduced the partial differential equation (16.12) to the geometric problem of constructing Λ_ϕ as a submanifold of Σ_p. To be able to use this approach we need to characterize, at least locally, those submanifolds Λ of $T^*\mathbb{R}^N$ which are of the form Λ_ϕ, i.e. are the graphs of differentials.

On $T^*\mathbb{R}^N$ there is a tautological 1-form, the contact form. By definition $T^*\mathbb{R}^N$ consists of pairs $(\overline{z}, \overline{\gamma})$ where $\overline{\gamma}$ is a 1-form at $\bar{z} \in \mathbb{R}^N$. Thus γ can be considered as an equivalence class of smooth functions near $\bar{z}, [f]$, where $f' \sim f$, i.e. $f'' \in [f]$ means precisely that $f - f' - f(\bar{z}) + f(\bar{z})$ vanishes quadratically at $\bar{z}$. The equivalence class can be identified with the vector of partial derivatives of f :

$$[f] = \sum_{i=0}^{n} \frac{\partial f}{\partial z_i}(z) dz_i \tag{16.16}$$

where $dz^i = [z^i]$. Then $\zeta_i = \frac{\partial f}{\partial z_i}(\bar{z})$ are the canonically dual coordinates of γ (dual to $z_0, \ldots, z_1$ that is.) Let $\pi : T^*\mathbb{R}^N \ni (\bar{z}, \bar{\gamma}) \longmapsto \bar{z} \in \mathbb{R}^N$ be the natural projection. We can use π to pull-back 1-forms:

$$\pi^* : T^*_{\bar{z}}\mathbb{R}^N \longrightarrow T^*_{(\bar{z},\bar{\gamma})}(T^*\mathbb{R}^N) \quad \forall\ (\bar{z}, \bar{\gamma}) \in T^*\mathbb{R}^N. \tag{16.17}$$

Then the tautological 1-form, α, on $T^*\mathbb{R}^N$ is fixed by the condition:

$$\alpha = \pi^*\bar{\gamma} \quad \text{ at } (\bar{z}, \bar{\gamma}). \tag{16.18}$$

Since, in canonically dual coordinates, $\bar{\gamma} = \sum\limits_{i=0}^{n} \zeta_i dz_i$

$$\alpha = \sum_{i=0}^{n} \zeta_i dz_i. \tag{16.19}$$

Clearly α is completely well-defined, independent of the choice of coordinates. Its exterior derivative is the 2-form

$$\omega = d\alpha = \sum_{i=0}^{n} d\zeta_i \wedge dz_i. \tag{16.20}$$

Using ω, the symplectic form on $T^*\mathbb{R}^N$, we can characterize the submanifolds Λ_ϕ.

PROPOSITION 16.43. *If $\Lambda \subset T^*\mathbb{R}^N$ is a $\mathcal{C}^\infty$ submanifold then, locally near $(\bar{z}, \bar{\zeta}) \in \Lambda$, $\Lambda = \Lambda_\phi$ for some $\phi \in \mathcal{C}^\infty(\Omega)$, $\Omega \ni \bar{z}$ open if and only if*

$$\dim \Lambda = N \tag{16.21}$$

$$\iota_\Lambda^*\omega = 0,\ \ \iota_\Lambda : \Lambda \hookrightarrow T^*\mathbb{R}^N \text{ being the inclusion} \tag{16.22}$$

$$\pi : \Lambda \longrightarrow \mathbb{R}^N \text{ is, near } (\bar{z}, \bar{\zeta}),\ \text{ a local diffeomorphism.} \tag{16.23}$$

In general a submanifold satisfying just conditions (16.21) *and* (16.22) *is said to be a Lagrangian submanifold of $T^*\mathbb{R}^N$.*

PROOF. First we have to check that these conditions hold for Λ_ϕ. Both (16.21) and (16.23) are immediate. For Λ_ϕ we know that the inverse to (16.23) is just the defining map

$$\Omega \ni z \overset{\Phi}{\rightarrow}\!\!\longmapsto (z, d_z\phi) \in \Lambda_\phi. \tag{16.24}$$

Consider the pull-back under Φ to $\Omega \ni \bar{z}$ of the 2-form ω. Since $\omega = d\alpha$,

$$\Phi^*\omega\big|_{\bar{z}} = d\Phi^*\alpha\big|_{\bar{z}}\,. \tag{16.25}$$

From (16.18), $\Phi^*\alpha = \Phi^*\pi^*\bar{\gamma} = d\phi$ at $\bar{\gamma} = d\phi$. Thus from (16.25), $\Phi^*\omega = d(d\phi) = 0$. This proves that (16.22) holds.

To prove the converse, suppose (16.21) – (16.23) hold ((16.21) is of course a consequence of (16.23)). From (16.23), $\Lambda = \{(z, \Xi(z))\}$ where $\Xi_j(z)$ are $\mathcal{C}^\infty$ functions on $\Omega \ni \bar{z}$. Then (16.22) requires that

$$d(\sum_{i=0}^{N} \Xi_i(z)dz_i) = 0 \text{ in } \Omega. \tag{16.26}$$

Poincaré's lemma asserts that, in a possibly smaller neighbourhood of $\bar{z}$,

$$\sum_{i=0}^{N} \Xi_i(z)dz_i = d\phi \tag{16.27}$$

for some $\mathcal{C}^\infty$ function ϕ. This just asserts that $\Lambda = \Lambda_\phi$ near $(\bar{z}, \bar{\gamma})$. □

Thus the problem of solving (16.12) has been reduced to constructing a suitable Lagrangian submanifold of Σ_p satisfying (16.23). Observe that Σ_p is a smooth hypersurface in $T^*\mathbb{R}^N\backslash 0$, where 0 stands for the zero section $\{(z, 0)\}$. Indeed

$$dp = 2\zeta_0 d\zeta_0 - d\sum_{i,j=1}^{n} g^{ij}(z)\zeta_i\zeta_j \neq 0 \tag{16.28}$$

if $\zeta_0 \neq 0$, so as $\zeta_0^2 = \Sigma g^{ij}(x)\zeta_i\zeta_j$ on Σ_p, $dp \neq 0$ on $\Sigma_p \cap \{\zeta \neq 0\}$.

As a smooth hypersurface in a symplectic manifold $(T^*\mathbb{R}^N\backslash 0)$, Σ_p has a well-defined 1-dimensional foliation, i.e. a 1-dimensional subbundle of $T\Sigma_p$. This subbundle $H_\Sigma \subseteq T\Sigma_p$ is called the Hamilton foliation and is defined by

$$v \in H_\Sigma \Longleftrightarrow \omega(v, w) = 0 \quad \forall\ w \in T\Sigma_p. \tag{16.29}$$

LEMMA 16.51. *Suppose p is $\mathcal{C}^\infty$ and $dp \neq 0$ on $(p = 0)$ in $T^*\mathbb{R}^N$ then the Hamilton foliation is spanned by the Hamilton vector field of p :*

$$H_p = \sum_{i=0}^{n}\left(\frac{\partial p}{\partial \zeta_i}\frac{\partial}{\partial z_i} - \frac{\partial p}{\partial z_i}\frac{\partial}{\partial \zeta_i}\right). \tag{16.30}$$

PROOF. The definition (16.30) of H_p can be written in the more intrinsic form

$$\omega(H_p, v) = -dp(v) \quad \forall\ v \in T(T^*\mathbb{R}^N). \tag{16.31}$$

Indeed, that (16.30) satisfies (16.31) follows by evaluating the left side of (16.31). Conversely if (16.31) holds then (16.30) follows by setting $v = \frac{\partial}{\partial z_i}$, and $v = \frac{\partial}{\partial \zeta_i}$, the basis vector fields.

From (16.31), by setting $v = H_p$,

$$0 = \omega(H_p, H_p) = -dp(H_p) \tag{16.32}$$

follows from the antisymmetry of ω and shows that H_p is tangent to Σ_p. Now suppose (16.29) holds. Since Σ_p has codimension one it follows that

$$\omega(v, w) = cdp(w) \quad \forall\ w \in T(T^*\mathbb{R}^*) \tag{16.33}$$

for some constant C. Then as before, $v = cH_p$. □

The most basic component of Hamilton-Jacobi theory is now easily derived:

LEMMA 16.52. *If $\Lambda \subset \Sigma$ is a Lagrangian submanifold contained in a hypersurface in a symplectic manifold then the Hamilton foliation of Σ is tangent to Λ.*

from -250 to 250, y from -200 to 250 .5pt ¡4pt¿ [.2,.67] from -200 0 to 200 0 ¡4pt¿ [.2,.67] from 0 -180 to 0 180 -150 -150 150 150 / from

FIGURE 1. Characteristic Lagrangians

PROOF. Consider the vector space, at $\lambda \in \Lambda, T_\lambda\Lambda + \mathbb{R}H_p = Q_\lambda$. This has dimension N, or $N+1$ as H_p is, or is not, tangent to Λ at λ. Thus we wish to show that $\dim Q_\lambda = N$. Now, by definition – (16.22) – and (16.31)

$$\omega|_{Q\times Q} \equiv 0. \tag{16.34}$$

Thus it suffices to show that ω can vanish on no subspace of dimension greater than N. This is an easy exercise in linear algebra; it follows from a simple diagonalization procedure. □

To construct Λ_ϕ we now proceed to 'integrate' the vector field H_p. Recall the initial condition for f

$$\phi|_{t=0} = \phi_0 = s - x \cdot \omega. \tag{16.35}$$

This implies that

$$\Lambda_\phi \cap \{t = 0\} = \{(0, x, \tau(x), \frac{\partial \phi_0}{\partial x})\} \tag{16.36}$$

for some smooth function $\tau(x)$. However we also want $\Lambda_\phi \subset \Sigma_p$, so

$$\tau^2 = \left|\frac{\partial \phi_0}{\partial x}\right|^2 = \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial \phi_0}{\partial x_i}\frac{\partial \phi_0}{\partial x_j}(x) \Longrightarrow \tau = \pm\left|\frac{\partial \phi_0}{\partial x}\right|. \tag{16.37}$$

These functions are smooth since $\partial\phi_0/\partial x \neq 0$. Thus there are precisely two choices for

$$\Lambda_\phi^\pm \cap \{t = 0\} = \left\{(0, x, \pm\left|\frac{\partial \phi_0}{\partial x}\right|, \frac{\partial \phi_0}{\partial x})\right\}. \tag{16.38}$$

This leads to the two solution of (16.12).

The vector field H_p is transversal to $\{t = 0\}$ on Σ_p. Thus we must have each integral curve of H_p through a point of $\Lambda_\phi^\pm \cap \{t = 0\}$ in $\Lambda_\phi^\pm$. This leads to two N-dimensional manifolds:

$$\Lambda_\phi^\pm = H_p\text{-flow-out of } \left\{(0, x, \pm\left|\frac{\partial \phi_0}{\partial x}\right|, \frac{\partial \phi_0}{\partial x})\right\}. \tag{16.39}$$

The theory of ordinary differential equations (including the smooth dependence of solutions on initial data) shows that $\Lambda_\phi^\pm$ are $\mathcal{C}^\infty$, at least for small t. In fact they are globally smooth, uniquely fixed by the maximal extensions of the integral curves of H_p of which they are unions.

PROPOSITION 16.44. *If* $\phi_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$ *has* $d_x\phi_0 \neq 0$ *then* (16.39) *leads to the two smooth Lagrangian submanifolds of* $T^*\mathbb{R}^N$ *contained in* Σ_p *and satisfying* (16.36).

PROOF. The uniqueness is clear, we need to show that $\Lambda_\phi^\pm$ defined by (16.39) are Lagrangian.

This is a direct consequence of an identity for the Lie derivative. If V is any $\mathcal{C}^\infty$ vector field then the Lie derivative L_V is a first order differential operator acting on each bundle of k-forms, defined by

$$L_V\gamma = (d \cdot i_V + i_V \cdot d)\gamma \tag{16.40}$$

where i_V is contraction with V, $i_V\gamma(V_1,\dots,V_k)=\gamma(V,V_1,\dots V_k)$ if γ is a $(k+1)$-form. Now, if $V=H_p$ is the Hamilton vector field of p then

$$L_{H_p}\omega = d\cdot i_{H_p}\omega + i_{H_p}d\omega = d(dp)=0, \tag{16.41}$$

using (16.31) – which just says $i_{H_p}\omega = dp$. This means that ω satisfies a first-order differential equation on Λ along the integral curves of H_p. From the uniqueness of the solution to the initial value problem for such a system it is enough to check that

$$\omega = 0 \text{ on } T_\lambda\Lambda^{\pm},\ \ \lambda\in\Lambda\cap\{t=0\}. \tag{16.42}$$

By definition of $\Lambda^{\pm}\cap\{t=0\}$, $T_\lambda\Lambda^{\pm}$ is spanned by $T_{\lambda'}\Lambda_0$ and H_p, where $\lambda=(0,\bar{x},\tau,\bar{\xi})$ if $\lambda'=(\bar{x},\bar{\xi})$. Certainly ω vanishes on $T_{\lambda'}\Lambda_0$, since this is Lagrangian in $T^*\mathbb{R}^n$. Thus (16.42) reduces to showing that $\omega(v,H_p)=0$ if $v\in T_{\lambda_0'}$, but again this follows from (16.31). This completes the proof that $\Lambda^{\pm}$ are Lagrangian. □

Finally note that $\Lambda^{\pm}_{\phi}$ satisfy the fibre-transversal condition (16.40), near any point in $\Lambda^{\pm}\cap\{t=0\}$ since H_p is transversal to $t=0$. Thus we have shown:

Proposition 16.45. *There is an open neighbourhood, Ω, of $t=0$ such that the problem* (16.12) *has precisely two $\mathcal{C}^\infty$ solutions in $\mathbb{R}^n\times(-\epsilon,\epsilon)$ for some $\epsilon>0$, $\phi^{\pm}$, given by the conditions*

$$\phi^{\pm}|_{t=0}=\phi_0,\quad \Lambda^{\pm}=\Lambda_{\phi^{\pm}}. \tag{16.43}$$

Proof. We have shown this locally for each s,ω and the proof obviously extends to give smoothness in s,ω locally. Since S^{n-1} is compact the fact that the solutions exists in the fixed neighbourhood $\mathbb{R}^n\times(-\epsilon,\epsilon)$ for all s reduces to understanding what happens for large s, and near infinity in $\mathbb{R}^n$. However s is merely an additive constant so $\phi_{\pm}(t,x,\omega,s)=\phi_{\pm}(t,x,\omega,0)+s$ and for $|x|\ge 2\rho$, $|t|\le\rho$, $\phi_{\pm}(t,x,\omega,s)=s-x\cdot\omega\pm t$. Thus the local uniqueness proves the semiglobal (global in $\mathbb{R}^n$, for short times) existence. □

The fact that these "phase functions" $\phi_{\pm}$ exists only for short times is real, and not just an artifice of our method of construction. We could (and in a sense shall) use the theory of conormal distributions to construct solutions to (16.6) for short times. Using these we can, as for the potential case, construct a forward fundamental solution (for small t) and hence show the existence of the wave 'group' $U(t)$, for $|t|<\epsilon$.

Indeed, once $U(t)$ has been constructed for $|t|<\epsilon$, satisfying $U(t)U(s)=U(t+s)$, whenever $|s|$, $|t|$, $|t+s|<\epsilon$, it can be extended uniquely as a group by setting $U(t)=U(\frac{t}{n})^n$. The question therefore arises, and is in any case fundamental to understanding the structure of the scattering matrix, as to what is the nature of the *global* solution to (16.6) (which can be constructed using $U(t)$).

Rather than follow this small-time approach we shall proceed directly to construct a global solution to (16.6). We have already noted that the Lagrangian $\Lambda^{\pm}_{\phi}$ exists globally. For short times we expect the solution to (16.6) to be conormal with respect to $\{\phi_{\pm}=0\}$. Consider the conormal bundles of these submanifolds

$$N^*\{\phi_{\pm}=0\}=\{(z,\tau d\phi_{\pm});\phi_{\pm}(z)=0,\tau\in\mathbb{R}\backslash 0\}\subset T^*\mathbb{R}^N\cap\{|t|<\epsilon\}. \tag{16.44}$$

Knowing $N^*\{\phi_{\pm}=0\}$ is completely equivalent to knowing $\{\phi_{\pm}=0\}$. Thus we can think of u, solving (16.6), as being associated to $N^*\{\phi_{\pm}=0\}$ in $|t|<\epsilon$.

The submanifolds

$$\tilde{\Lambda}^{\pm} = H_p\text{-flow-out of } N^*\{\phi_{\pm} = 0\} \tag{16.45}$$

are contained in the homogeneous extensions of the $\Lambda^{\pm}$ (i.e. $\lambda \in \tilde{\Lambda}^{\pm} \implies \tau\lambda \in \Lambda^{\pm}$ for some $\tau \neq 0$).

EXERCISE 16.13. Exercise Show that $\tilde{\Lambda}^{\pm}$ are Lagrangian submanifolds of $T^*\mathbb{R}^N \backslash 0$.

Thus to construct u globally we proceed to develop the theory of *Lagrangian distributions*. From (16.45) we see that $\tilde{\Lambda}^{\pm} \cap \{|t| < \epsilon\} = N^*\{\phi_{\pm} = 0\}$. Thus the theory of Lagrangian distributions should be (and is) a natural extension of the theory of conormal distributions.

CHAPTER 17

Parametrization of conic Lagrangians

In examining the push-forward of conormal distributions, associated to a hypersurface, we studied the tangency of a hypersurface to the leaves of a fibration. In $M = \mathbb{R}^n_x \times \mathbb{R}^N_\theta$ we consider a hypersurface (which need not be closed)

$$H = \{\phi = 0\} \ \phi \in \mathcal{C}^\infty(M), \ d\phi \neq 0 \text{ on } H. \tag{17.1}$$

Thinking of $\pi : M \longrightarrow \mathbb{R}^n_x$ as a fibration means that we shall allow only coordinate transformations which are fibre-preserving on M, iė

$$(x, \theta) \longmapsto (X(x), \Theta(x, \theta)). \tag{17.2}$$

For the push-forward of a conormal distribution associated to H to be conormal on $\mathbb{R}^n$ we needed a condition of simple tangency of H to the fibres (or rather simple tangency of the fibres to H!) Locally this condition takes the form

(17.3) ϕ has only non-degenerate critical points on each fibre.

Then the set of such fibre-critical points

$$C_H = \{(x, \theta) \in M; \phi(x, \theta) = 0, \ \partial_{\theta_j} \phi(x, \theta) = 0, j = 1, \dots, N\} \tag{17.4}$$

is a submanifold of M of codimension $N + 1$ and the projection

$$\pi : C_H \longrightarrow \mathbb{R}^n \tag{17.5}$$

is locally the embedding of C_H as a hypersurface in $\mathbb{R}^n$. We required (but could do without) the condition that (17.5) should be a global embedding.

As promised then, we now consider the weakening of (17.3) to:

(17.6) At C_H the differentials $d\phi, d[\partial_{\theta_j} \phi]$ $j = 1, \dots, N$ are linearly independent.

This still implies, via the implicit function theorem, that C_H is a $\mathcal{C}^\infty$ submanifold of M of codimension $N + 1$. However it need not even be locally embedded in $\mathbb{R}^n$ by the projection (17.5). Nevertheless we wish still to think of C_H as a 'singular submanifold' of $\mathbb{R}^n$. The way to do this is to consider the conormal bundle of this singular manifold. If (17.3) holds then the conormal bundle of $C_H \hookrightarrow \mathbb{R}^n$ can be written in the following way

$$\begin{gathered} N^* \pi(C_H) = \\ \{(x, \xi); \xi = \tau d_x \phi(x, \theta), \tau \in \mathbb{R}, \phi(x, \theta) = 0, \partial_{\theta_j} \phi(x, \theta) = 0 \text{ for some } \theta.\} \end{gathered} \tag{17.7}$$

Indeed this just follows from the observation that, locally, the surface

$$\partial_\theta \phi(x, \theta) = 0 \tag{17.8}$$

is transversal to π and the restriction of ϕ to it pulls back under the inverse of π to a defining function for $\pi(C_H)$. In general we just use (17.7) as the definition of the singular submanifold.

LEMMA 17.53. *If $H \subset M$ is a $\mathcal{C}^\infty$ hypersurface, ϕ being a defining function for it, with* (17.6) *satisfied then*

$$\Lambda_H = \{(x, \tau d_x\phi(x,\theta); (x,\theta) \in C_H, \tau \neq 0\} \subset T^*\mathbb{R}^n \setminus 0 \tag{17.9}$$

is a $\mathcal{C}^\infty$ conic (i.e. homogeneous) Lagrangian submanifold.

PROOF. Consider the map

$$C_H \ni (x,\theta) \longmapsto (x, d_x\phi(x,\theta)) \in T^*\mathbb{R}^n \setminus 0. \tag{17.10}$$

Any tangent vector annihilated by the differential of this map must be tangent to the fibres, since the map is trivial in the x variables. A tangent vector, $v = a_1\partial_{\theta_1} + \cdots + a_N\partial_{\theta_N}$, to the fibres annihilated by the differential of (17.10) represents a dependence relation

$$\sum_{j=1}^N a_j \frac{\partial^2\phi}{\partial\theta_j\partial x_i} = 0 \Longleftrightarrow \sum_{j=1}^N a_j d_x \partial_{\theta_j}\phi = 0. \tag{17.11}$$

To be tangent to C_H the vector must satisfy in addition

$$\sum_{j=1}^N a_j \frac{\partial^2\phi}{\partial\theta_i\partial\theta_j} = 0. \tag{17.12}$$

Thus such a tangent vector represents a linear relation amongst the $d\partial_{\theta_j}\phi$ at C_H, so must vanish by the assumption (17.6). Hence the differential of (17.10) is injective, so embeds C_H as a smooth submanifold of dimension $n-1$ in $T^*\mathbb{R}^n \setminus 0$. Moreover the $\mathbb{R}^+$ action (multiplication by scalars on the fibres) on $T^*\mathbb{R}^n$ cannot be tangent to the range of (17.10), since this would mean

$$\sum_{j=1}^N a_j d_x\left(\frac{\partial\phi}{\partial\theta_j}\right) = \tau d_x\phi \tag{17.13}$$

which, as above, would lead to a dependence relation between the $d\partial_{\theta_j}\phi$ and $d\phi$ violating the full force of (17.6). Thus Λ_H is a $\mathcal{C}^\infty$ conic submanifold of dimension n in $T^*\mathbb{R}^n \setminus 0$.

It remains to show that the symplectic form vanishes on Λ_H. To do this consider again the definition (17.9). This can be rewritten in terms of the space of fibre-normals. The submanifold

$$S = \{(x,\theta,\xi,0) \in T^*M; \xi \neq 0\} \tag{17.14}$$

is just the set of lifts to TM of non-vanishing differentials on $\mathbb{R}^n$, i.e. the range of

$$\pi^* : T^*_{\pi(m)}\mathbb{R}^n \longrightarrow T^*_m M. \tag{17.15}$$

Let $\pi_S : S \longrightarrow T^*\mathbb{R}^n$ be the left inverse of π^*, the obvious projection. Then (17.9) is just

$$\pi_S : [N^*C_H \cap S] \longrightarrow \Lambda_H \text{ is a diffeomorphism .} \tag{17.16}$$

Of course we know that N^*C_H is a Lagrangian submanifold of T^*M. Certainly therefore the symplectic form

$$\omega_M = \sum_{i=1}^n dx_i \wedge d\xi_i + \sum_{j=1}^N d\theta_j \wedge dt_j \tag{17.17}$$

vanishes on $N^*C_H \cap S$. Moreover, since $S = \{t = 0\}$ we have

$$\pi_S^* \omega = \iota_S^* \omega_M \quad \text{(on } M) \tag{17.18}$$

where ω is the symplectic form on $T^*\mathbb{R}^n$ and ι_S is the inclusion of S in T^*M. This proves that ω vanishes on Λ_H, since this is equivalent to the vanishing of ω_M on $N^*H \cap S$. □

This basic lemma shows that we can construct conic Lagrangian submanifolds by pushing forward hypersurfaces. We shall say that $H \subset M$ is a parametrization of Λ near $\lambda \in \Lambda$ if

$$\Lambda \cap \Omega = \Lambda_H \cap \Omega \text{ for some neighbourhood } \Omega \text{ of } \lambda \text{ in } T^*\mathbb{R}^n \setminus 0. \tag{17.19}$$

Before discussing the very important result that any conic Lagrangian submanifold of $T^*\mathbb{R}^n \setminus 0$ has a parametrization near each of its points we note, for orientation, a couple of simple examples of parametrization.

Example 17.0. The function $\phi = x_1 \cos(\theta) + x_2 \sin(\theta)$ on $\mathbb{R}^2 \times \mathbb{R}$ parametrizes (repeatedly) the Lagrangian submanifold of $T^*\mathbb{R}^2 \setminus 0$ which is just the fibre above the origin $T_0^*\mathbb{R}^2 \setminus 0$.

Exercise 17.14. Show that the hypersurface

$$H = \{t^3 - 3ty + 2x = 0\} \subset \mathbb{R}^2 \times \mathbb{R} \tag{17.20}$$

parametrizes a $\mathcal{C}^\infty$ conic Lagrangian submanifold of $T^*\mathbb{R}^2$ which is the conormal bundle of the 'cusp' $y^3 = x^2$ except over the origin. Convince yourself that Λ_H is just the closure in $T^*\mathbb{R}^2 \setminus 0$ of the conormal bundle to the regular part of this singular curve. Can you find another (really different, i.e. not locally transformable into the cusp by a coordinate transformation) singular curve with this remarkable property – that the closure of the conormal bundle to the smooth part is actually a smooth conic Lagrangian??

The main point of parametrizations is that they always exist

Proposition 17.46. *If $\Lambda \subset T^*\mathbb{R}^n \setminus 0$ is a $\mathcal{C}^\infty$ conic Lagrangian submanifold and $\bar\lambda \in \Lambda$ then for some $H \subset \mathbb{R}^n \times \mathbb{R}^N$, with $\pi(\bar\lambda) \in H$, a $\mathcal{C}^\infty$ hypersurface satisfying* (17.6) *with $N < n$ there is a neighbourhood Ω of $\bar\lambda$ in $T^*\mathbb{R}^n \setminus 0$ such that*

$$\Lambda \cap \Omega = \Lambda_H \cap \Omega. \tag{17.21}$$

Proof. Consider the projection of the tangent space to Λ at $\bar\lambda$ to $\mathbb{R}^n$:

$$\pi_*[T_{\bar\lambda}\Lambda] \subset T_{\pi(\bar\lambda)}\mathbb{R}^n. \tag{17.22}$$

We can make a linear transformation so that $\pi(\bar\lambda) = 0$ and the projection in (17.22) is spanned by ∂_{x_i} for $i = 1, \dots, n - N - 1$, so defining N. (Notice that the fact that Λ is conic means that the space in (17.22) cannot have dimension larger than $n - 1$.) In terms of the canonically dual coordinates (x, ξ) in $T^*\mathbb{R}^n$ this means that there are tangent vectors of the form

$$v_i = \partial_{x_i} + \sum_{j=1}^{N} \gamma_{ij}\partial_{\xi_j} \in T_{\bar\lambda}\Lambda, \; i = 1, \dots, n - N - 1 \tag{17.23}$$

and these can be extended to a basis of $T_{\bar\lambda}\Lambda$ by adding vectors of the form

$$w_k = \sum_{j=1}^{N} \gamma'_{kj}\partial_{\xi_j} \;\; k = n-N,\dots,n. \tag{17.24}$$

It follows that the matrix

$$\Gamma' = \left(\gamma'_{kj}\right)_{k,j=n-N,\dots,n} \text{ is invertible.} \tag{17.25}$$

Indeed, this invertibility is equivalent to the non-existence of a (non-zero) tangent vector to Λ at $\bar\lambda$ of the form

$$w = \sum_{j<n-N} g_j\partial_{\xi_j}. \tag{17.26}$$

Pairing with the symplectic form gives

$$\omega(w, v_j) = g_j,\;\; j = 1,\dots,n-N-1 \tag{17.27}$$

so if w in (17.26) is in $T_{\bar\lambda}\Lambda$ it must vanish, Λ, being Lagrangian.

The conclusion of this computation is that the functions

$$x_1,\dots,x_{n-N-1},\xi_{n-N},\dots,\xi_n \text{ give local coordinates on } \Lambda \text{ near } \bar\lambda. \tag{17.28}$$

Another way of expressing this is that there are $\mathcal{C}^\infty$ functions

$$\begin{aligned} X''(x',\xi'') &= (X_{n-N}(x',\xi''),\dots,X_n(x',\xi''))\\ \Xi'(x',\xi'') &= (\Xi_1(x',\xi''),\dots,\Xi_{n-N-1}(x',\xi''))\\ \text{s.t. } \Lambda &= \{x'' = X''(x',\xi''), \xi' = \Xi'(x',\xi'')\} \text{ near } \bar\lambda. \end{aligned} \tag{17.29}$$

Here $x' = (x_1,\dots,x_{n-N-1})$ and $\xi'' = (\xi_{n-N},\dots,\xi_n)$ etc. The fact that Λ is conic means that X'' and Ξ' are respectively homogeneous of degrees zero and one in ξ''.

We can easily express the condition that Λ be Lagrangian in terms of the local coordinates x', ξ''. Namely

$$\omega = \sum_{i=1}^{n-N-1} dx_i \wedge d\Xi_i + \sum_{j=n-N}^{n} dX_j \wedge d\xi_j = 0. \tag{17.30}$$

This in turn just means

$$d\left[-\sum_{i=1}^{n-N-1} \Xi_i dx_i + \sum_{j=n-N}^{n} X_j dx_j\right] = 0. \tag{17.31}$$

By Poincaré's Lemma this means that locally, near any point, there is a $\mathcal{C}^\infty$ function such that

$$-\sum_{i=1}^{n-N-1} \Xi_i dx_i + \sum_{j=n-N}^{n} X_j dx_j = d\Phi(x',\xi'') \tag{17.32}$$

Moreover Φ is unique up to a constant and so can be assumed to be homogeneous of degree one in ξ''. Since Λ is conic the vector $\xi\cdot\partial_\xi$ is in $T_{\bar\lambda}\Lambda$. We know from the discussion around (17.26) that if $\xi = \bar\xi$ at $\bar\lambda$ then at least one of the $\bar\xi_j$, for $j \ge n-N$

must be non-zero. Making a linear change of coordinates, just amongst the x'', we may as well assume that $\bar{\xi}'' = (0, \dots, 0, 1) \in \mathbb{R}^{n-N}$. Now consider the function

$$\phi(x,\theta) = x_n + \sum_{j=1}^{N} x_{n-N-1+j}\theta_j - \Phi(x', \theta, 1). \tag{17.33}$$

We proceed to show that the hypersurface defined by $\phi = 0$ parametrizes Λ near $\bar{\lambda}$.

First we should check the non-degeneracy condition (17.6). However the independence of the differentials is clear since, at $(x, \theta) = (0, 0)$,

$$\begin{gathered} d\phi = dx_n,\ d[\partial_{\theta_j}\phi] = dx_{j+n-N-1} \\ \mod \{dx', d\theta\}\ j = 1, \dots, N. \end{gathered} \tag{17.34}$$

This proves the non-degeneracy, so ϕ parametrizes some Lagrangian. For future reference, note that

$$\frac{\partial^2 \Phi}{\partial \theta^2}(0,0) = 0. \tag{17.35}$$

From (17.32), the definition of Φ,

$$\Xi_j = -\frac{\partial \Phi}{\partial x_j}\Phi(x', \xi''),\ X_\ell = \frac{\partial \Phi}{\partial \xi_\ell} x', \xi''),\ j < n - N,\ \ell \geq n - N. \tag{17.36}$$

Since Φ is homogeneous of degree 1 in ξ'',

$$\phi = \tilde{\phi}(x, \theta, 1),\ \tilde{\phi}(x, \xi'') = x''.\xi'' - \Phi(x', \xi''). \tag{17.37}$$

Thus, using Euler's identity for homogeneous functions

$$\phi = x_n - \frac{\partial \Phi}{\partial \xi_n}(x', \theta, 1) + \sum_{j=1}^{N} (x_{j+n-N-1} - \frac{\partial \Phi}{\partial \Theta_j}(x', \theta, 1))\theta_j. \tag{17.38}$$

It follows, using (17.36), that

$$\phi = 0,\ \frac{\partial \phi}{\partial \theta_j} = 0, j = 1, \dots, N \Longrightarrow x'' = \frac{\partial \Phi}{\partial \xi''}(x', \theta, 1) = X''(x', \theta, 1). \tag{17.39}$$

Now Λ_H is the cone over the image of

$$p_H : C_H \ni (x, \theta) \longmapsto (x, d_x\phi(x, \theta) = (x, -\frac{\partial \Phi}{\partial x'}, \theta, 1). \tag{17.40}$$

From (17.39) and (17.36) and the homogeneity of Λ it follows that

$$\Lambda_H \subset \Lambda \text{ near } \bar{\lambda}. \tag{17.41}$$

Since the manifolds have the same dimension they must be equal near $\bar{\lambda}$. This completes the proof of the proposition. □

The existence of a parametrization near each point of a conic Lagrangian allows us to define the space of Lagrangian distributions associated with it. We do this by recalling that if the hypersurface $H \subset \mathbb{R}^n \times \mathbb{R}^N$ should satisfy (17.3), and (17.5) is a global embedding, then push-forward gives a map

$$\pi_* : I_c^m(\mathbb{R}^{n+N}, H) \longrightarrow I^{m-N/4}(\mathbb{R}^n, \pi(H)). \tag{17.42}$$

For conormal distributions associated to a hypersurface $G \subset \mathbb{R}^n$ we use the alternative notation

$$I^m(\mathbb{R}^n, G) \equiv I^m(\mathbb{R}^n, N^*G). \tag{17.43}$$

Thus we think of G as represented by its conormal bundle as a conic Lagrangian submanifold $N^*G \subset T^*\mathbb{R}^n\setminus 0$. Then (17.42) becomes

$$\pi_* : I_c^{m+N/4}(\mathbb{R}^{n+N}, N^*H) \longrightarrow I^m(\mathbb{R}^n, \Lambda_H) \tag{17.44}$$

since $\Lambda_H = N^*G$ by the definition of parametrization of Lagrangians.

We shall use (17.44) as the basis of the definition of Lagrangian distributions associated to any $\mathcal{C}^\infty$ conic Lagrangian.

Definition 17.12. If $\Lambda \subset T^*\mathbb{R}^n\setminus 0$ is a $\mathcal{C}^\infty$ conic Lagrangian then $I_c^m(\mathbb{R}^n, \Lambda) \subset \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ consists precisely of those distributions which can be written as finite sums

$$I_c^m(\mathbb{R}^n, \Lambda) \ni u \Longleftrightarrow u = \sum_{k=1}^{p} (\pi_p)_*(u_p) \tag{17.45}$$

where each $u_p \in I^{n+N_p/4}(\mathbb{R}^{n+N_p}, H_p)$ for a (globally embedded) hypersurface $H_p \subset \mathbb{R}^{n+N_p}$ satisfying (17.6) parametrizing a Lagrangian $\Lambda_p \subset T^*\mathbb{R}^n\setminus 0$ such that for some open set $\Omega_p \subset \mathbb{R}^n$

$$\Lambda_p \cap \pi^{-1}(\Omega_p) = \Lambda \cap \pi^{-1}(\Omega_p) \text{ and } \operatorname{supp}(u_p) \subset \pi_p^{-1}(\Omega_p). \tag{17.46}$$

Since H_p is assumed to be a globally embedded hypersurface we do not want to assume that it parametrizes Λ globally, so (17.46) is just there to ensure that it does parametrize Λ in a neighbourhood of the support of u_p. Since $I^m(\mathbb{R}^n, \Lambda)$ consists of sums of push-forwards it is clear that it is a linear space. To handle elements reasonably it is of primary importance to show that we can represent any Lagrangian distribution in terms of a sum of push-forwards from any parametrizations which cover Λ.

There is an obvious case in which the push-forward $(\pi_1)_*(u_1)$ of a conormal distribution from one parametrization $H_1 \subset \mathbb{R}^{n+N_1}$ can be expressed in the form $(\pi_2)_*(u_2)$ with u_2 conormal to another parametrization $H_2 \subset \mathbb{R}^{n+N_2}$. Namely if there exists a fibre preserving diffeomorphism

$$\chi(x,\theta) = (x, \Theta(x,\theta)),\ \chi : \mathbb{R}^n \times \mathbb{R}^{N_1} \longleftrightarrow \mathbb{R}^n \times \mathbb{R}^{N_2},\ \chi(H_1) = H_2. \tag{17.47}$$

Of course this implies that $N_1 = N_2 = N$. Indeed if (17.47) holds then

$$\chi^* : I^{m+N/4}(\mathbb{R}^{n+N}, H_2) \longleftrightarrow I^{m+N/4}(\mathbb{R}^{n+N}, H_1), \tag{17.48}$$

using the coordinate-invariance of conormal distributions. We can therefore write the push-forward of u_2 as

$$\begin{gathered}(\pi_2)_*(u_2) = \int_{\mathbb{R}^N} u_2(x,\theta')d\theta' = \int_{\mathbb{R}^N} u_1(x,\theta)\left|\frac{\partial\Theta(x,\theta)}{\partial\theta}\right| d\theta \\ \text{if } u_1(x,\theta) = u_2(x,\Theta(x,\theta)) = \chi^*(u_2).\end{gathered} \tag{17.49}$$

Thus push-forward from $I^{m+N/4}(\mathbb{R}^{n+N}, H_1)$ gives exactly the same distributions as push-forward from $I^{m+N/4}(\mathbb{R}^{n+N}, H_2)$. Of course this same argument, (17.49), works if χ is only defined as a local diffeomorphism and u_2 has support in the range. We shall therefore say that $H_1 \subset \mathbb{R}^{n+N}$ and $H_2 \subset \mathbb{R}^{n+N}$ are *equivalent* parametrizations in open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+N}$ if there is a fibre-preserving diffeomorphism $\chi_1 : \Omega_1 \longrightarrow \Omega_2$ such that $\chi(H_1 \cup \Omega_1) = H_2 \cup \Omega_2$.

The main technical problem is therefore to decide when two parametrizations are equivalent in this sense. We shall say that they are equivalent near $(\bar{x}, \bar{\theta}_1) \in C_{H_1}$

and $(\bar{x}, \bar{\theta}_2) \in C_{H_2}$ if they are equivalent in some neighbourhoods of these points by a fibre-preserving diffeomorphism such that

$$\chi(\bar{x}, \bar{\theta}_1) = (\bar{x}, \bar{\theta}_2). \tag{17.50}$$

We can relatively easily extract necessary conditions for local equivalence. Of course we must have the equality of fibre dimensions. Secondly we must have equality of the Lagrangians parametrized, locally:

$$\begin{gathered} \text{If } H_1 \text{ and } H_2 \text{ are equivalent parametrizations near } (\bar{x}, \bar{\theta}_1) \text{ and } (\bar{x}, \bar{\theta}_2) \\ \text{then } \Lambda_1 = \Lambda_2 \text{ near } p_1(\bar{x}, \bar{\theta}_1) = p_2(\bar{x}, \bar{\theta}_2). \end{gathered} \tag{17.51}$$

Here of course Λ_i are the Lagrangians parametrized by H_i and the p_i are the parametrization maps (17.40). Indeed if ϕ_2 defines H_2 then $\phi_1 = \chi^* \phi_2$ defines H_2. Thus

$$\partial_\theta \phi_1(x, \theta) = \partial_\theta \phi_2(x, \Theta(x, \theta)) = \frac{\partial \Theta}{\partial \theta} \partial_{\theta'} \phi_2(x, \Theta(x, \theta)). \tag{17.52}$$

This shows that χ maps the critical set of one onto the critical set of the other

$$\chi : C_1 \longleftrightarrow C_2 \tag{17.53}$$

and

$$(x, \theta) \in C_1 \Longrightarrow p_1(x\theta) = (x, d_x \phi_1(x, \theta)) \tag{17.54}$$

$$= (x, d_x \phi_2(x, \Theta(x, \theta))) = p_2(\chi(x, \theta)). \tag{17.55}$$

Thus we get a commutative diagram

There is however another necessary condition for equivalence following from (17.52). Differentiating again with respect to θ and using the fact that $\partial_\theta \phi = 0$ on the critical set we find that

$$\frac{\partial^2 \phi_1}{\partial \theta_i \partial \theta_j}(x, \theta) = \sum_{k,\ell} \frac{\partial \Theta_k}{\partial \theta_i} \frac{\partial \Theta_\ell}{\partial \theta_j} \frac{\partial^2 \phi_2}{\partial \theta'_k \partial \theta'_\ell} \phi_2(x, \Theta(x, \theta)) \text{ on } C_1. \tag{17.56}$$

This means that at each point of C_1, and in particular at the base point $(\bar{x}, \bar{\theta}_1)$, the fibre-Hessian matrices satisfy

$$\frac{\partial^2 \phi}{\partial \theta^2}(\bar{x}, \bar{\theta}_1) = J \frac{\partial^2 \phi_2}{\partial \theta^2}(\bar{x}, \bar{\theta}_2) J^t \tag{17.57}$$

where J is the fibre differential of χ and J^t is its transpose. It follows that

$$\frac{\partial^2 \phi}{\partial \theta^2}(\bar{x}, \bar{\theta}_1) \text{ and } \frac{\partial^2 \phi_2}{\partial \theta^2}(\bar{x}, \bar{\theta}_2) \text{ have the same signature.} \tag{17.58}$$

Here the signature is the number of positive minus the number of negative eigenvalues of these real symmetric matrices. Of course they also have the same rank, but this is already a consequence of (17.55) or (**??**).

The next major result is that these conditions are sufficient for local equivalence.

PROPOSITION 17.47. *If $H_i \subset \mathbb{R}^{n+N}$, $i = 1, 2$, are two $\mathcal{C}^\infty$ hypersurfaces near $(\bar{x}, \bar{\theta}_i) \in C_i$ both satisfying* (17.6)*, near their respectively base points, then provided* (17.55) *and* (17.58) *hold they are locally equivalent parametrizations.*

Proof. We prove this important result in three stages. First we show that a fibre-preserving transformation χ can be chosen mapping the critical set C_H to $C_{H'}$. Then, using (17.58), we show that after such a transformation there is an homotopy H_t with $H_0 = H$, $H_1 = H'$ parametrizing Λ for each t with (17.6) holding throughout. This homotopy is then used to construct χ via an homotopy of fibre-preserving transformations, given by integration of a (t-dependent) vector field.

First we make a translation in θ' so that the base points are the same, $(\bar{x}, \bar{\theta}) = (\bar{x}, \bar{\theta}')$. We can also make a linear transformation, using (17.58), so that

$$\frac{\partial^2 \phi}{\partial\theta_i \partial\theta_j}\phi(\bar{x}, \bar{\theta}) = \frac{\partial^2 \tilde{\phi}}{\partial\theta_i \partial\theta_j}\phi(\bar{x}, \bar{\theta}) = \pm\delta_{ij},\ 1 \le i, j \le N' \le N,\ 0 \text{ otherwise.} \tag{17.59}$$

It is now convenient to regard θ and θ' as the same coordinates in a fixed space $\mathbb{R}^{n+N}$. We wish to make a fibre-preserving transformation mapping C_H to $C_{H'}$. Consider the map, with x as a parameter

$$\mathbb{R} \times \mathbb{R}^N \ni (\tau, \theta) \longmapsto \tau\frac{\partial\phi}{\partial x_1}, \dots, \tau\frac{\partial\phi}{\partial x_n}, \tau\frac{\partial\phi}{\partial\theta_1}, \dots, \tau\frac{\partial\phi}{\partial\theta_N}. \tag{17.60}$$

By (17.59) and the non-degeneracy assumption on ϕ, the Jacobian matrix at $x = \bar{x}$, $(\tau, \theta) = (1, \bar{\theta})$ has rank $N + 1$ since it is

$$\begin{pmatrix} \frac{\partial\phi}{\partial x_1}, \dots, \frac{\partial\phi}{\partial\theta_N} \\ \frac{\partial^2\phi}{\partial x_1 \partial\theta_1}, \dots, \frac{\partial^2\phi}{\partial\theta_N \partial\theta_1} \\ \\ \frac{\partial^2\phi}{\partial x_1 \partial\theta_N}, \dots, \frac{\partial^2\phi}{\partial\theta_N \partial\theta_N} \end{pmatrix} \tag{17.61}$$

In fact, after relabelling the x-coordinates, we can assume that

$$(\tau, \theta) \longmapsto \tau\frac{\partial\phi}{\partial x_1}, \dots, \tau\frac{\partial\phi}{\partial x_{N-N'+1}}, \tau\frac{\partial\phi}{\partial\theta_1}, \dots, \tau\frac{\partial\phi}{\partial\theta_{N'}} \tag{17.62}$$

is a local diffeomorphism with N' as in (17.59). Since $\tilde{\phi}$ parametrizes the same Lagrangian near $(\bar{x}, \bar{\theta})$ it follows that the same map, (17.62), with ϕ replaced by $\tilde{\phi}$, is also a local diffeomorphism, $\tilde{F}$. The composite map $\tilde{F}^{-1}F$ is of the form

$$\tilde{F}_x^{-1}F_x(\tau, \theta) = (\tau\tau'(x, \theta), \Theta(x, \theta)) \tag{17.63}$$

since (17.62) is obviously homogeneous of degree one in τ.

The map $(x, \theta) \longmapsto (x, \Theta(x, \theta))$ reduces C_H to $C_{H'}$. To see this observe that if $(x, \theta) \in C_H$ and $(X, \theta') \in C'_H$ parametrize the same points, up to a multiple, in Λ, $\lambda = (x, d_x\phi(x, \theta) = (x, \tau' d_x\tilde{\phi}(x, \theta'))$ then from (17.62)

$$F_x(x, \theta) = \tau'\tilde{F}_x(x, \theta') \Longleftrightarrow \tilde{F}_x^{-1}F_x(1, \theta) = (\tau', \theta') \tag{17.64}$$

so $\theta' = \Theta(x, \theta)$ showing that C_H is mapped to $C_{H'}$.

Even more is true. If $(x, \theta') = (x, \Theta(x, \theta))$ then

$$\tau(x, \theta)\frac{\partial\tilde{\phi}}{\partial\theta_i}(x, \Theta(x, \theta)) = \frac{\partial\phi}{\partial\theta_i},\ i = 1, \dots, N'. \tag{17.65}$$

Differentiating with respect to θ_i for $i = 1, \dots, N'$ and using (17.59) we see that

$$\tau\sum_{\ell=1}^{N'}\frac{\partial^2\tilde{\phi}}{\partial\theta_i\partial\theta_\ell}\frac{\partial\Theta_\ell}{\partial\theta_j} = \frac{\partial^2\phi}{\partial\theta_i\partial\theta_j} \Longrightarrow \bar{\tau}\frac{\partial\Theta_\ell}{\partial\theta_j}(\bar{x}, \bar{\theta}) = \delta_{\ell j},\ \ell, j =, \dots, N' \tag{17.66}$$

for some $\bar{\tau} > 0$. Thus we conclude that the transformed function

$$\theta'(x,\theta) = \tilde{\phi}(x, \Theta(x,\theta)) \tag{17.67}$$

still has the property (17.62) and moreover $C_H = C_{H'}$.

This completes the first part of the proof, and also the second part if we set

$$\phi_t(x,\theta) = (1-t)\phi(x,\theta) + t\phi'(x,\theta),\ 0 \le t \le 1. \tag{17.68}$$

Certainly ϕ_t is always fibre-critical on C_H and the non-degeneracy follows from the fact that (17.62) holds throughout the homotopy.

The final step in the proof is to construct a 1-parameter family of (fibre-preserving) diffeomorphisms, χ_t, such that

$$\chi_0 = \mathrm{Id} \ \text{ and } \chi_t^*(\alpha_t\phi_t) = \phi,\ \alpha_t > 0,\ \alpha_0 = 1. \tag{17.69}$$

Differentiating (17.69) with respect to t gives

$$\frac{d}{dt}\chi_t^*(\alpha_t\phi_t) = \chi_t^*\left[\alpha_t\dot{\phi}_t + (\dot{\alpha}_t + V_t\alpha_t)\phi_t + \alpha_t V_t\phi_t\right] = 0 \tag{17.70}$$

where the dot denotes t-differentiation. Here V_t is a smooth vector field which is determined by, and through integration determines, χ_t :

(17.71)

$$\frac{d}{dt}\Theta_i(t,x,\theta) = V_t^i(t,x,\Theta(t,x,\theta)),\ \chi_t(x,\theta) = (x,\Theta(t,x,\theta)),\ \Theta(0,x,\theta) = 0.$$

Of course the basis of this "homotopy method" for attacking conjugation problems is that to satisfy (17.70) we just need

$$\dot{\phi}_t = -V_t\phi_t + \beta_t\phi_t,\ \beta_t = -\alpha_t^{-1}[\dot{\alpha}_t + V_t\alpha_t] \tag{17.72}$$

in which χ_t no longer appears explicitly.

By (17.68) we know that $d\phi_t/dt = 0$ on C_H and the non-degeneracy of ϕ_t means that for some functions V^i and β

$$\frac{d\phi_t}{dt} = -\sum_{j=1}^{N} V^i(t,x,\theta)\frac{\partial\phi_t}{\partial\theta_j} + \beta(t,x,\theta)\phi_t. \tag{17.73}$$

This fixes the vector field $V_t = V^1\partial/\partial\theta_1 + \cdots + V^N\partial/\partial\theta_N$. To ensure (17.72) we just need to pick α as the solution of

$$\frac{d\alpha_t}{dt} + V_t\alpha_t = -\beta\alpha_t,\ \alpha|_{t=0} = 1 \tag{17.74}$$

which just involves integration along the vector field $d/dt + V_t$. Thus we have constructed V so that (17.70) holds. This completes the proof of the proposition. □

CHAPTER 18

Symbol calculus for Lagrangian distributions

CHAPTER 19

The scattering relation

Bibliography

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