An introduction to microlocal analysis with applications to inverse problems, summer 2016

Exercise Problems, all lectures
Return your solutions to Teemu Saksala by 11th of Sebtember at 23:59 by e-mail teemu.saksala@helsinki.fi. In order to get the one credit for the exercises, you should solve atleast 5 of the following problems.

Please let me know, if you find any mistakes etc.

Notations:

- $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-indes. $|\alpha| := \sum_{k=1}^n \alpha_k$
- $\partial_{x_k} := \frac{\partial}{\partial x_k}$ is the k^{th} partial derivative with respect to Cartesian coordinates and $D_k = -i\partial_{x_k}$
- $\partial^\alpha = \prod_{k=1}^n \partial_{x_k}^{\alpha_k}$ and $D^\alpha = \prod_{k=1}^n (-i\partial_{x_k})^{\alpha_k} = (-i)^{|\alpha|} \prod_{k=1}^n \partial^{\alpha_k}$

Problem 1. *Let $w \in \mathbb{R}^n$, $\|w\| = 1$ and $s \in \mathbb{R}$. We denote the Hyperplane*

$$H_{w,s} = \{x \in \mathbb{R}^n : x \cdot w = s\}.$$

Let $dx = d_{x_1} \wedge d_{x_2} \wedge \ldots \wedge d_{x_n}$ be the volume form of $\mathbb{R}^n$. Then hyperplane $H_{w,s}$ has a natural volume form dH given by formula

$$dH = (N \lrcorner dx)|_{H_{w,s}},$$

where N is a unit normal of $H_{s,w}$ and $\lrcorner$ stands for interior multiplication. (See [6]).

Show that the equation

$$dx|_{H_{s,w}} = ds \wedge dH$$

is valid. Here ds should be considered to be the differential of the mapping $x \mapsto x \cdot w$.

Problem 2. *Recall that set $U \subset (S^{n-1} \times \mathbb{R})$ is open if and only if for every $p \in U$ there exists a set $p \in (V \times (a, b)) \subset U$, where $V \subset S^{n-1}$ is open.*

Let $f \in C_0^\infty(\mathbb{R}^n)$. We define the Radon transform of f by formula

$$Rf(w, s) = \int_{x \cdot w = s} f(x) dH, \ (w, s) \in (S^{n-1} \times \mathbb{R}).$$

Show that $R : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(S^{n-1} \times \mathbb{R})$ is well defined, linear and continuous.

Problem 3. *Recall that the formal transpose R^t of Radon transform is defined by L^2-duality*

$$(Rf, g)_{L^2(S^{n-1}\times\mathbb{R})} = (f, R^t g)_{L^2(\mathbb{R}^n)}, \ \ f \in C_0^\infty(\mathbb{R}^n)\,, g \in C_0^\infty(S^{n-1} \times \mathbb{R}).$$

Then it holds that

$$R^t : C_0^\infty(S^{n-1} \times \mathbb{R}) \to C^\infty(\mathbb{R}^n), \ \ R^t g(x) = \int_{S^{n-1}} g(x \cdot w, w) dw.$$

Compute the normal operator $R^t R$ and show that

$$(R^t R) f = c_n\, \phi * f,$$

where c_n is a dimensional constant and $\phi(x) = \frac{1}{\|x\|}$.

Problem 4. *Show that*

$$\mathcal{F}(R^tRf)(\xi) = c_n \frac{\widehat{f}(\xi)}{\|\xi\|^{n-1}}$$

Problem 5. *Find $f \in C_0^\infty(\mathbb{R}^\times S^{n-1})$ such that $R^t f$ is not compactly supported.*

Problem 6. *Recall the Radon inversion formula (RIF) for test functions is*

$$f = c_n(-\Delta)^{\frac{n-1}{2}} R^tRf, \; f \in C_0^\infty(\mathbb{R}^n),$$

where for $\alpha \in \mathbb{R}$ such that $-n < \alpha$ we define

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \|\xi\|^\alpha \widehat{f}(\xi) d\xi.$$

Show that (RIF) is also valid for any compactly supported distribution. I.e. show

$$u = c_n(-\Delta)^{\frac{n-1}{2}} R^tRu \;\; u \in \mathcal{E}'(\mathbb{R}^n)$$

Problem 7. *Recall that the wave front set of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is defined by negation as $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ is not in WFu, if there exists a neighborhood $(U \times V) \subset \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ of (x_0, ξ_0) such that for all $\varphi \in C_0^\infty(U)$, $\xi \in V$ and $k \in \mathbb{N}$ holds*

$$|\mathcal{F}(\varphi u)(t\xi)| \le C_n |1 + t|^{-k}, \; t > 0.$$

Let $n = 2$ and denote by χ the characteristic function of an open unit disc $B(0, 1) \subset \mathbb{R}^2$. Prove that

$$WF\chi = \{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\}) : \|x\| = 1, \; \xi \parallel x\}.$$

Problem 8. *Let χ be the characteristic function of unit square $Q := [0, 1] \times [0, 1] \subset \mathbb{R}^2$ prove that*

$$WF\chi = \{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\}) : x \in \partial Q, \xi \parallel x\} \bigcup \{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\}) : x_i = \{0, 1\}, \; \xi \in \mathbb{R}^2 \backslash \{0\}\}.$$

I.e. at the corner points every direction is in the wavefront set.

Problem 9. *Let $F \subset \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ be closed and conic. Show that there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ such that*

$$WFu = F.$$

Problem 10. *Let $k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We define a linear operator.*

$$K : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \; Kf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy.$$

Then the adjoint of K with respect to L^2 innerproduct is

$$K^t f(y) = \int_{\mathbb{R}^n} k(x, y) f(x) dx, \; f \in C_0^\infty(\mathbb{R}^n)$$

Prove that for any $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\langle u, K^t\varphi \rangle = \int_{\mathbb{R}^n} \langle u, k(x, \cdot) \rangle \varphi(x) dx$$

Problem 11. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and $k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that*

$$\langle u, k(x, \cdot) \rangle \in C^\infty(\mathbb{R}^n).$$

Problem 12. *Recall the Schwartz kernel theorem. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be open sets. Let $A : C_0^\infty(X) \to \mathcal{D}'(Y)$ be linear and continuous. Then there exists a unique $k_A \in \mathcal{D}'(X \times Y)$ such that*

$$\langle A\varphi, \psi \rangle = k_A(\varphi \otimes \psi),\ \varphi \in C_0^\infty(X),\ \psi \in C_0^\infty(Y).$$

Here the tensor product $(\varphi \otimes \psi)(x, y) := \varphi(x)\psi(y)$.

If $a \in C^\infty(X \times Y)$, it determines naturally the operator $A : C_0^\infty(X) \to \mathcal{D}'(Y)$

$$A\varphi(\psi) = \int_X \int_Y a(x, y)\varphi(x)\overline{\psi(y)}dxdy$$

Let $X = Y \subset \mathbb{R}^n$. Consider a partial differential operator

$$A = \sum_{|\alpha| \leq k} a_\alpha D^\alpha,\ a_\alpha \in C^\infty(X).$$

Show that the Schwartz kernel of operator A is

$$k_A(x, y) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha \delta(x - y).$$

Problem 13. *Let $m \in \mathbb{N}$ and $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. We define the Schwartz kernel k_p of p as*

$$\langle k_p, \varphi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \frac{p(x, \xi)}{(1 + |\xi|^2)^M} (I + \Delta_y)^M \varphi(y)\ dxdyd\xi,\ \ \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

Show that k_p is well defined and independent of M, if $M \geq \frac{m+n}{2}$

Problem 14. *Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be s.t. $\eta(x)1$ when $\|x\| \leq 1$. Show that*

$$k_p = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \eta(\epsilon\xi) e^{i(x-y)\cdot\xi} p(x, \xi), d\xi$$

where $k_p \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by equation (1).

Problem 15. *Let $A, B \in \Psi^m(\mathbb{R}^n)$. show that the Schwartz kernel k_{AB} of composition operator AB satisfies*

$$k_{AB}(x, y) = \int_{\mathbb{R}^n} k_A(x, z) k_B(z, y) dz,$$

when ever right hand side is well defined.

Problem 16. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(x) = 1$, if $\|x\| \leq 1$. Let $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Show that function*

$$F(x, y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \frac{1 - \chi(x - y)}{\|x - y\|^{2M}} \Delta_\xi^M p(x, \xi)\ d\xi \in C^k(\mathbb{R}^n \times \mathbb{R}^n),$$

for all $k \in \mathbb{N}$ and is independent of M, if M is large enough.

Show that

$$k_{\widetilde{A}}(x, y) := \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \chi(x - y) p(x, \xi)\ d\xi$$

is a Schwartz kernel of some $\widetilde{A} \in \Psi^m(\mathbb{R}^n)$.

Problem 17. *Let $A \in \Psi^m(\mathbb{R}^n)$. Show that there is a extension*

$$\widetilde{A} : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

of A that is linear and continuous.

Suppose that $A \in \Psi^m(\mathbb{R}^n)$ is properly supported. Show that there is a linear and continuous extension

$$B : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

of $\widetilde{A}$.

Recall that pseudo differential operator A is properly supported, if the Schwartz kernel k_A is properly supported in $\mathbb{R}^n \times \mathbb{R}^n$ i.e.

$$supp k_A \subset \mathbb{R}^n \times \mathbb{R}^n$$

is proper. A set $X \subset \mathbb{R}^n \times \mathbb{R}^n$ is proper, if for all compact $K \subset \mathbb{R}^n$ the sets

$$\pi_x(\pi_y^{-1} K \cap X) \text{ and } \pi_y(\pi_x^{-1} K \cap X)$$

are compact in $\mathbb{R}^n$. Here $\pi_y(x, y) = y$ and $\pi_x(x, y) = x$.

Problem 18. *Show that for any $A \in \Psi^m(\mathbb{R}^n)$ holds*

$$WF(Au) \subset WFu, \text{ for any } u \in \mathcal{E}'(\mathbb{R}^n).$$

You can use the fact

$$singsupp(Au) \subset singsupp(u), \text{ for any } u \in \mathcal{E}'(\mathbb{R}^n).$$

References

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