Inverse problems course, Exercise 6 (for the week starting on March 13, 2017)
University of Helsinki
Department of Mathematics and Statistics
Samuli Siltanen, Markus Juvonen, Minh Mach, Santeri Kaupinmäki and Alexander Meaney
Related book sections (Mueller & Siltanen 2012): Sections 5 and 6.

Theoretical exercises:

T1. **Normal equations for generalized Tikhonov regularization.** Take $\alpha > 0$ and consider the following Tikhonov-type functional containing two different regularizers:

$$\mathbf{f}_\alpha = \underset{\mathbf{f}\in\mathbb{R}^n}{\arg\min}\{\|A\mathbf{f}-\mathbf{m}\|_2^2 + \alpha\|L_1\mathbf{f}\|_2^2 + \alpha\|L_2\mathbf{f}\|_2^2\}.$$

Here L_1 is a $k_1 \times n$ matrix and L_2 is a $k_2 \times n$ matrix. Recall that if we take L_1 to be the identity matrix and L_2 a zero matrix, then $\mathbf{f}_\alpha$ solves the normal equations

$$(A^TA + \alpha I)\mathbf{f}_\alpha = A^T\mathbf{m}.$$

Use a computation similar to the one in Section 5.2 of the textbook (Mueller & Siltanen 2012) to show that, in the general case,

$$(A^TA + \alpha L_1^TL_1 + \alpha L_2^TL_2)\mathbf{f}_\alpha = A^T\mathbf{m}.$$

Matlab exercises:

M1. **Generalized Tikhonov regularization for 2D tomography.** Use the tomographic phantom with three rectangles with resolution 128×128. Simulate crime-free tomographic data $\widetilde{\mathbf{m}}$ with 31 angles, and construct the measurement matrix A. Take horizontal and vertical difference matrices L_H and L_V from the Matlab file *tomo07_TV_comp.m*.

(a) **Stacked-form style.** Write

$$A' = \begin{bmatrix} A \\ \sqrt{\alpha} L_H \\ \sqrt{\alpha} L_V \end{bmatrix}, \qquad \widetilde{\mathbf{m}}' = \begin{bmatrix} \widetilde{\mathbf{m}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{1}$$

Compute the regularized reconstruction as

$$\mathbf{f}_\alpha = A' \setminus \widetilde{\mathbf{m}}'. \tag{2}$$

(b) **Conjugate-gradient style.** Solve the normal equations

$$(A^T A + \alpha L_H^T L_H + \alpha L_V^T L_V)\mathbf{f}_\alpha = A^T \mathbf{m}$$

using the iterative conjugate gradient method described in Subsection 5.5.2 of the textbook (Mueller & Siltanen 2012). Do you get the same result than in (a)?

(c) Using either one of the two approaches, find the α giving the smallest relative error. (Note that there is no nonnegativity constraint included!)

M2. **Total variation and L^1 norm regularization for 2D tomography.** Use exactly the same problem setup than in Problem M1.

(a) Use Matlab file *tomo07_TV_comp.m* to compute the Total Variation regularized reconstruction

$$\mathbf{f}_\alpha = \underset{\mathbf{f} \in \mathbb{R}^n_+}{\arg\min} \{ \|A\mathbf{f} - \mathbf{m}\|_2^2 + \alpha \|L_H \mathbf{f}\|_1 + \alpha \|L_V \mathbf{f}\|_1 \}.$$

Here $\mathbf{f} \in \mathbb{R}^n_+$ means that we include the nonnegativity constraint $\mathbf{f}_j \geq 0$. How small relative error can you get by varying α?

(Actually, the above is called *anisotropic* Total Variation regularization. The *isotropic* version would use $\alpha\sqrt{\|L_H\mathbf{f}\|_2^2 + \|L_V\mathbf{f}\|_2^2}$ instead of $\alpha\|L_H\mathbf{f}\|_1 + \alpha\|L_V\mathbf{f}\|_1$, but that would be harder to put into a format suitable for `quadprog`. However, there are many solution methods in the literature for the isotropic case as well.)

(a) Modify Matlab file *tomo07_TV_comp.m* to compute

$$\mathbf{f}_\alpha = \underset{\mathbf{f} \in \mathbb{R}^n_+}{\arg\min} \{ \|A\mathbf{f} - \mathbf{m}\|_2^2 + \alpha \|\mathbf{f}\|_1 \}.$$

How small relative error can you get by varying α?