Inverse problems course, Exercise 1 (for the week starting on January 30, 2017)
University of Helsinki
Department of Mathematics and Statistics
Samuli Siltanen, Minh Mach, Santeri Kaupinmäki and Alexander Meaney
Related book sections (Mueller & Siltanen 2012): 2.1.1, 2.1.2, 3.5 and 4.2.

Theoretical exercises:

T1. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{for } -0.1 \leq x \leq 0.1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the function $g * g$ analytically (by hand), where

$$(g * g)(x) = \int_{-\infty}^{\infty} g(x')g(x - x')\, dx'.$$

Outside which interval $[a, b] \subset \mathbb{R}$ is $(g * g)(x) = 0$?

T2. Assume that the $n \times n$ matrix U is orthogonal: $UU^T = I = U^T U$.

(a) Show that $\|U^T y\| = \|y\|$ for any $y \in \mathbb{R}^n$.

(b) Take $n = 2$, let U be as above and let $x, y \in \mathbb{R}^2$. Show that the angle between the vectors x and y is the same than the angle between the vectors Ux and Uy.

T3. Let A be a real-valued $n \times n$ matrix.

(a) Show that the matrix $A^T A$ is symmetric.

(b) Show that if λ is an eigenvalue of $A^T A$, then $\lambda \geq 0$.

T4. Let A be a real-valued $n \times n$ matrix. Recall from basic linear algebra that a symmetric matrix can be diagonalized and its eigenvectors can be chosen to be orthonormal. Denote the eigenvalues of $A^T A$ by

$$d_1^2 \geq d_2^2 \geq \cdots \geq d_r^2 > d_{r+1}^2 = d_{r+1}^2 = \cdots = d_n^2 = 0,$$

and the corresponding orthonormal eigenvectors by $V^{(1)}, V^{(2)}, \ldots, V^{(n)}$. Insert the eigenvectors as columns to a matrix called V. Also, write $V = [V_1\, V_2]$ with

$$V_1 = [V^{(1)}\, V^{(2)} \cdots V^{(r)}], \qquad V_2 = [V^{(r+1)}\, V^{(r+2)} \cdots V^{(n)}].$$

Then

$$V^T A^T A V = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix},$$

where the $r \times r$ matrix Σ is defined by $\Sigma^2 = \text{diag}(d_1^2, \ldots, d_r^2)$. Here $V_1^T A^T A V_1 = \Sigma^2$. **Show that** $AV_2 = 0$. Now define a $n \times r$ matrix U_1 by $U_1 = AV_1\Sigma^{-1}$. **Show that** $U_1^T U_1 = I$. Therefore the columns of U_1 are orthonormal. **Show that** we can define an orthonormal $n \times n$ matrix in the form $U = [U_1\, U_2]$. **Finally, derive the SVD by showing that**

$$U^T A V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}.$$

Hint: use the block forms of the matrices.

Matlab exercises:

M1. Let the point spread function $p \in \mathbb{R}^5$ and the vector $\mathbf{f} \in \mathbb{R}^{16}$ be defined by

$$\begin{aligned} p &= [p_{-2}, p_{-1}, p_0, p_1, p_2]^T = [\frac{1}{16}, \frac{3}{16}, \frac{1}{2}, \frac{3}{16}, \frac{1}{16},]^T, \\ \mathbf{f} &= [\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6, \mathbf{f}_7, \mathbf{f}_8, \mathbf{f}_9, \mathbf{f}_{10}, \mathbf{f}_{11}, \mathbf{f}_{12}, \mathbf{f}_{13}, \mathbf{f}_{14}, \mathbf{f}_{15}, \mathbf{f}_{16}]^T \\ &= [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T. \end{aligned}$$

Note the non-standard indexing of the vector p. Define the discrete convolution vector $(p * \mathbf{f}) \in \mathbb{R}^{16}$ by

$$(p * \mathbf{f})_j = \sum_{\ell=-2}^{2} p_\ell \mathbf{f}_{j-\ell}, \qquad 1 \leq j \leq 16, \tag{1}$$

where we use the following definition for out-of-bounds indices:

$$\mathbf{f}_{j-\ell} = 0 \quad \text{for } j - \ell < 1 \text{ and } j - \ell > 16. \tag{2}$$

Boundary condition (2) is called *zero extension*.

(a) Use a for-loop to calculate the vector $p * \mathbf{f}$ straight from definition (1) using the boundary condition (2).

(b) Use the command `convmtx` for building a convolution matrix A. Calculate the vector $p * \mathbf{f}$ as $A\mathbf{f}$. Check that you get the same result than in (a).

(b) Use the command `conv2` to calculate the vector $p*\mathbf{f}$. Check that you get the same result than in (a).

M2. Periodic boundary conditions are defined by

$$\begin{array}{llll} \mathbf{f}_0 = \mathbf{f}_{16}, & \mathbf{f}_{-1} = \mathbf{f}_{15}, & \mathbf{f}_{-2} = \mathbf{f}_{13}, & \ldots, \\ \mathbf{f}_{17} = \mathbf{f}_1, & \mathbf{f}_{18} = \mathbf{f}_2, & \mathbf{f}_{19} = \mathbf{f}_3, & \ldots \end{array} \tag{3}$$

(a) Use a for-loop to calculate the vector $p * \mathbf{f}$ straight from definition (1) using the boundary condition (3).

(b) Use the command `convmtx` for building a convolution matrix A. Modify the convolution matrix so that it implements the periodic boundary conditions (3). Calculate the vector $p * \mathbf{f}$ as $A\mathbf{f}$. Check that you get the same result than in (a).

(b) Use Fast Fourier Transform (FFT) to calculate the vector $p * \mathbf{f}$ with the periodic boundary conditions (3). In principle this approach takes the simple form

$$\texttt{ifft}(\texttt{fft}(p) \cdot \texttt{fft}(\mathbf{f})), \tag{4}$$

where $\cdot$ stands for element-wise vector product. However, in (4) the vectors p and $\mathbf{f}$ have to have the same length, so you need to "zero-pad" vector p so that it has 16 elements. In the zero-padding process you need to be careful with the location of the centerpoint of the PSF. Studying the command `fftshift` may help you.

Check that you get the same result than in (a).