Finite model theory

Problems 10

Tuesday 22.11.2016

1. Show that existential second-order logic Σ^1_1 is closed under $\vee$ and $\forall x$ in the following sense: if $\varphi, \psi \in \Sigma^1_1$, then there is $\theta \in \Sigma^1_1$ such that for all $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi \vee \psi \Leftrightarrow \mathfrak{A} \models \theta$$

and

$$\mathfrak{A} \models \forall x \varphi \Leftrightarrow \mathfrak{A} \models \theta.$$

2. Let $\varphi \in \mathrm{FO}[\tau]$, where $\tau = \tau_1 \cup \{<\}$ and τ_1 is finite. Sentence φ is *order-invariant* if for all τ_1-models $\mathfrak{A}$ and all linear orderings $<, <'$ of $\mathrm{Dom}(\mathfrak{A})$:

$$\langle \mathfrak{A}, < \rangle \models \varphi \Leftrightarrow \langle \mathfrak{A}, <' \rangle \models \varphi.$$

Show the following:

i) Give an example of a first-order sentence φ which is not order-invariant.

ii) Let $\tau_1 = \{U, V, E\}$, where U, V are unary and E is binary. Show that there is an order-invariant sentence $\varphi \in \mathrm{FO}[\tau]$ that is not equivalent to any $\varphi \in \mathrm{FO}[\tau_1]$. (Hint: consider models of the form $\mathfrak{A}^{II}$ (see page 76 of the lecture notes), where V encodes $\mathcal{P}(U)$ and E encodes the set membership relation between elements of U and V. You may assume that in FO it is not possible to express that $|U|$ is even.)

3. Let τ be a finite relational vocabulary, and K a class of finite τ models. Let $< \notin \tau$ and let $K_<$ be the following class of $\tau \cup \{<\}$ models:

$$K_< = \{\langle \mathfrak{A}, < \rangle \mid \mathfrak{A} \in K \text{ and } < \text{ is an ordering of } \mathrm{Dom}(\mathfrak{A})\}.$$

Let $\mathcal{L}$ be a logic and C a complexity class. We say that $\mathcal{L}$ *strongly captures* C if for all τ and classes K of finite τ-models:

$$K_< \in C \Leftrightarrow K = \mathrm{Mod}(\varphi), \text{ for some } \varphi \in \mathcal{L}[\tau].$$

Show that IFP does not strongly capture PTIME.

4. Show that Σ^1_1 strongly captures NPTIME.